A version of Cartan's Theorem A for coherent sheaves on real affine varieties

Tomasz Kowalczyk

Jagiellonian University in Kraków

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Theorem 1 (Serre)

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X will be a non-singular real affine variety with structure sheaf $\mathcal{O}_X$.

Definition 2

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is called coherent if there exists a finite Zariski open covering $\{U_i\}_{i=1}^n$ of X such that for every U_i there is an exact sequence of sheaves

$$\mathcal{O}_X^{p_i}|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{q_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0.$$

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4. regulous geometry (Fichou, Huisman, Mangolte, Monnier)
5. regulous geometry over Henselian valued fields (Nowak)

Example 1

Let $P = X^2(X-1)^2 + Y^2 \in \mathbb{R}[X,Y]$*. The polynomial* P *is irreducible and has only two zeros* $c_1 = (0,0)$ *and* $c_2 = (1,0)$ *in* $\mathbb{R}^2$*. Put* $U_i = \mathbb{R}^2 \setminus \{c_i\}$*. The transition function*

$$g_{2,1} : U_1 \cap U_2 \to GL(1,\mathbb{R}) = \mathbb{R}^*$$

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defines a vector bundle of rank 1 over $\mathbb{R}^2$. *Global sections can be described as a pair* (s_1, s_2) *where* $s_i : U_i \to \mathbb{R}$ *are regular functions and* $g_{2,1}s_1 = s_2$. *Set* $s_i = \frac{f_i}{h_i}$ *where* f_i, h_i *are relatively prime polynomials. Then* $Pf_1 h_2 = f_2 h_1$. *Since* P *does not divide* h_2 *we obtain that* $f_2 = \lambda P f_1$ *and* $h_1 = \lambda^{-1} h_2$, *where* $\lambda \in \mathbb{R}^*$. *This shows that every algebraic global section of this bundle vanishes at* c_1.

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Definition 3

We say that a regular function $g : X \to \mathbb{R}$ on a non-singular real algebraic variety of dimension d is a simple normal crossing if in a neighbourhood of each point $a \in X$ one has

$$g(x) = u(x)x^{\alpha} = u(x)x_1^{\alpha_1}x_2^{\alpha_2}\dots x_d^{\alpha_d}$$

where $u(x)$ is a unit at a, $\alpha \in \mathbb{N}^d$ and $x = (x_1, x_2, \dots, x_d)$ are local coordinates near a, i.e. $x_1, x_2, \dots, x_d \in \mathcal{O}_{a,X}$ is a regular system of parameters of the local ring $\mathcal{O}_{a,X}$.

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Theorem 4

Let $f_1, f_2, \dots, f_k$ be regular functions on X. Then there exists a multi-blowup $\sigma : \widetilde{X} \to X$ such that $\sigma^* f_1, \sigma^* f_2, \dots, \sigma^* f_k$ are simple normal crossings.

Lemma 5

Let X be a non-singular real affine variety and $U = X \setminus \{Q = 0\}$ a Zariski open subset of X. Every regular function f on U can be written in the form $f = \frac{g}{P}$ where g, P are global regular functions on X and $V(P) \subset V(Q)$.

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Lemma 6

Let X be a non-singular real affine variety, Q a regular function on X and $U := X \setminus \{Q = 0\}$. Then for any $f \in \mathcal{O}_X(U)$ there exists a multi-blowup $\sigma : \widetilde{X} \to X$ and a positive integer N such that $(Q^N f)^\sigma$ can be extended to a global regular function, i.e. $(Q^N f)^\sigma \in \mathcal{O}_{\widetilde{X}}(\widetilde{X})$.

If the function f is of the form $f = \frac{g}{P}$ as a consequence of proof we get that

$$(Q^N)^\sigma \in P^\sigma \mathcal{O}_{\widetilde{X}}(\widetilde{X})$$

.

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Lemma 7

Let X be a non-singular real affine variety, $\mathcal{F}$ a coherent sheaf on X. For any $Q \in \mathcal{O}_X(X)$ and a section $s \in \mathcal{F}(X)$ such that $s|_U = 0$ with $U = X \setminus \{Q = 0\}$, there exists a multi-blowup $\sigma : \widetilde{X} \to X$ and a positive integer N such that $(Q^N)^\sigma \sigma^* s = 0$ in $\sigma^* \mathcal{F}(\widetilde{X})$.

Proof of Lemma 7

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2) $s|_{U_i} = \psi_i(t_i)$ for some $t_i \in \mathcal{O}_X^{q_i}(U_i)$.

Put

$$\mathrm{Rel}\,(t_i, \phi_i(e_1), \ldots, \phi_i(e_p); \mathcal{O}_X(U_i)) :=$$

$$= \left\{ (q, q_1, \ldots, q_p) \in \mathcal{O}_X(U_i)^{p+1} : qt_i + \sum_{j=1}^{p} q_j \phi_i(e_j) = 0 \right\}$$

and

Proof of Lemma 7

$$I_i := \left\{ q \in \mathcal{O}_X(U_i) : \exists q_1, \ldots, q_p \in \mathcal{O}_X(U_i) : qt_i + \sum_{j=1}^{p} q_j \phi_i(e_j) = 0 \right\} =$$

$$= \pi_1 \left(\mathrm{Rel} \left(t_i, \phi_i(e_1), \ldots, \phi_i(e_p); \mathcal{O}_X(U_i) \right) \right) \subset \mathcal{O}_X(U_i)$$

for each $i = 1, 2, \ldots, n$; here π_1 is the natural projection onto the first factor. Then, for every $x \in U_i$, we have

$$\mathrm{Rel} \left(t_i(x), \phi_i(e_1)(x), \ldots, \phi_i(e_p)(x); \mathcal{O}_{x,X} \right) =$$

$$\mathrm{Rel} \left(t_i, \phi_i(e_1), \ldots, \phi_i(e_p); \mathcal{O}_X(U_i) \right) \cdot \mathcal{O}_{x,X} \subset \mathcal{O}_{x,X}^{p+1}$$

because modules of relations commute with flat base change.

Proof of Lemma 7

Therefore

$$I_i \cdot \mathcal{O}_{x,X} = \{q_x \in \mathcal{O}_{x,X} : \exists q_{1x}, \ldots, q_{px} : q_x t_i(x) + \sum_{j=1}^{p} q_{jx}\phi(e_j)(x) = 0\}$$

and thus $1 \in I_i \cdot \mathcal{O}_{x,X}$ for every $x \in U_i \cap U$. Hence we get

$$U_i \cap V(I_i) \subset U_i \cap V(Q).$$

Clearly, there exists $p_i \in I_i$ such that $V(p_i) = V(I_i)$. Each p_i may be written in the form

$$p_i = \frac{P_i}{R_i} \text{ with } P_i, R_i \in \mathcal{O}_X(X) \text{ and } V(R_i) \cap U_i = \emptyset$$

for $i = 1, 2, \ldots, n$. Hence

$$V(p_i) \cap U_i = V(P_i) \cap U_i$$

and by Lemma 6 there exist a multi-blowup $\sigma : \widetilde{X} \to X$ and a positive integer N such that

Proof of Lemma 7

$$(Q^N)^\sigma|_{U_i^\sigma} \in P_i^\sigma|_{U_i^\sigma} \cdot \mathcal{O}_{\widetilde{X}}(U_i^\sigma), \quad i = 1, \ldots, n.$$

As p_i and P_i differ only by a unit on U_i, we get

$$(Q^N)^\sigma|_{U_i^\sigma} \in p_i^\sigma|_{U_i^\sigma} \cdot \mathcal{O}_{\widetilde{X}}(U_i^\sigma) \quad i = 1, \ldots, n.$$

Therefore

$$(Q^N)^\sigma|_{U_i^\sigma} \cdot \sigma^*(t_i) \in p_i^\sigma|_{U_i^\sigma} \cdot \sigma^*(t_i) \cdot \mathcal{O}_{\widetilde{X}}(U_i^\sigma) =$$
$$= \sigma^*(p_i|_{U_i} t_i) \cdot \mathcal{O}_{\widetilde{X}}(U_i^\sigma) \subset \sigma^*(\phi_i(\mathcal{O}_X^p(U_i))) = \phi_i^\sigma(\mathcal{O}_{\widetilde{X}}^p(U_i^\sigma)),$$

whence,

$$(Q^N)^\sigma|_{U_i^\sigma} \cdot \sigma^* s|_{U_i^\sigma} = \psi_i^\sigma((Q^N)^\sigma \sigma^* t_i) = 0|_{U_i^\sigma}.$$

This finishes the proof.

Let $f : X \to Y$ be a morphism of real algebraic varieties.

Lemma 8

If $\mathcal{G}$ is of finite type or coherent sheaf of $\mathcal{O}_Y$-Modules generated by sections $s_1, s_2, \ldots, s_k \in \mathcal{G}(Y)$, then the pull-back $f^*\mathcal{G}$ is generated by the pull-back $f^*s_1, f^*s_2, \ldots, f^*s_k \in (f^*\mathcal{G})(X)$.

Lemma 9

Let $\mathcal{F}$ be a coherent sheaf on a non-singular real affine variety X with local presentations

$$\mathcal{O}_X^p|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{q_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0 \quad i = 1, 2, \ldots, n$$

on a finite Zariski open covering $\{U_1, U_2, \ldots U_n\}$. Consider a finite family of Zariski open sets

$$V_j = X \setminus \{Q_j = 0\}, \quad j = 1, 2, \ldots, m$$

where Q_j are regular functions on X, and sections $s_j \in \mathcal{F}(V_j)$. Assume that every V_j is contained in $U_{i(j)}$ for some $i(j) = 1, 2, \ldots, n$ and that for each s_j there is a section $t_j \in \mathcal{O}_X^{q_{i(j)}}(V_j)$ such that $\psi_{i(j)}(V_j)(t_j) = s_j$. Then there exists a positive integer N and a multi-blowup $\sigma : \widetilde{X} \to X$ such that every section $(Q_j^N)^\sigma \sigma^* s_j \;\; j = 1, 2, \ldots, m$ extends to a global section on $\widetilde{X}$.

Proof of Lemma 9

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$$t_{jil} = \frac{t_{jil1}}{t_{jil2}}, \quad t_{jil1}, t_{jil2} \in \mathcal{O}_X(X)$$

and

$$V(t_{jil2}) \cap U_i \subset V(Q_j) \cap U_i.$$

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Using Lemma 6 we can find a positive integer N_1 and a multi-blowup $\tau_1 : X_1 \to X$ such that

$$(t_{jil}(Q_j)^{N_1})^{\tau_1} \in \mathcal{O}_{X_1}(U_i^{\tau_1}) \ \text{ for all } \ j, i, l.$$

Now define $\widetilde{s_{ji}} := \psi_i^{\tau_1}((t_{jil}(Q_j)^{N_1})^{\tau_1})$.

Proof of Lemma 9

Then for any two distinct indices i_0, i_1 we have

$$(\widetilde{s_{ji_0}} - \widetilde{s_{ji_1}})|_{U_{i_0}^{\tau_1} \cap U_{i_1}^{\tau_1}} \in (\tau_1^* \mathcal{F})(U_{i_0}^{\tau_1} \cap U_{i_1}^{\tau_1})$$

and

$$(\widetilde{s_{ji_0}} - \widetilde{s_{ji_1}})|_{U_{i_0}^{\tau_1} \cap U_{i_1}^{\tau_1} \cap V_j^{\tau_1}} = 0.$$

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By Lemma 7 we can find a multi-blowup $\tau_2 : \widetilde{X} \to X_1$ and a positive integer N_2 such that

$$(\tau_2^* \widetilde{s_{ji_0}} (Q_j^{N_1+N_2})^{\tau_1 \circ \tau_2} - \tau_2^* \widetilde{s_{ji_1}} (Q_j^{N_1+N_2})^{\tau_1 \circ \tau_2})|_{U_{i_0}^{\tau_1 \circ \tau_2} \cap U_{i_1}^{\tau_1 \circ \tau_2}} = 0.$$

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Considering all distinct pairs of indices, we can assume that the differences as above vanish for all those pairs. Therefore the sections

$$(\tau_2^* \widetilde{s_{ji}} (Q_j^{N_1+N_2})^{\tau_1 \circ \tau_2})|_{U_i^{\tau_1 \circ \tau_2}}, \quad i = 1, 2, \ldots, n,$$

glue together to a global section on $\widetilde{X}$. Thus $\sigma := \tau_1 \circ \tau_2 : \widetilde{X} \to X$ is the multi-blowup we are looking for.

Lemma 10

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-Modules of finite type and let $s_1, s_2, \ldots, s_k \in \mathcal{F}(U)$ be sections of $\mathcal{F}$ on a neighbourhood U of a point $a \in X$. If $s_{1a}, s_{2a}, \ldots, s_{ka}$ generate $\mathcal{F}_a$, then $s_{1x}, s_{2x}, \ldots, s_{kx}$ generate $\mathcal{F}_x$ for all x sufficiently close to a.

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Assume that, under the above assumptions, $U := X \setminus \{Q = 0\}$ with some $Q \in \mathcal{O}_X(X)$. Then the sections $Q^n s_1, Q^n s_2, \dots, Q^n s_k$ generate every stalk sufficiently close to a because the function Q is invertible in $\mathcal{O}_{a,X}$ for every $a \in U$. Now we are ready to prove the main theorem.

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Theorem 11

Let $\mathcal{F}$ be a coherent sheaf on a non-singular real affine variety X. Then there exist a multi-blowup $\sigma : \widetilde{X} \to X$ and finitely many global sections $s_1, s_2, \ldots, s_k$ on $\widetilde{X}$ which generate every stalk $(\sigma^* \mathcal{F})_y$, $y \in \widetilde{X}$.

Proof of Theorem 11

Consider a finite Zariski open covering $\{U_1, U_2, \ldots, U_n\}$ of X with local presentation of the sheaf $\mathcal{F}$

$$\mathcal{O}_X^p|_{U_i} \xrightarrow{\phi_i} \mathcal{O}_X^{q_i}|_{U_i} \xrightarrow{\psi_i} \mathcal{F}|_{U_i} \to 0.$$

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By Lemma 11, for any point $a \in X$ there are finitely many sections

$$s_{a1}, s_{a2}, \ldots, s_{am_a} \in \mathcal{F}(V_a), \quad m_a \in \mathbb{N},$$

on a Zariski open neighbourhood V_a of a, contained in U_i for some $i = 1, 2, \ldots, n$, which generate $\mathcal{F}$ over V_a.

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on a Zariski open neighbourhood V_a of a, contained in U_i for some $i = 1, 2, \dots, n$, which generate $\mathcal{F}$ over V_a. After shrinking V_a, we can also assume that $s_{ak} = \psi_i(t_{ak})$ for some $t_{ak} \in \mathcal{O}_X^{q_i}(V_a)$, $k = 1, 2, \dots, m_a$.

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$$s_{jk} = s_{a_j k} \quad \text{and} \quad t_{jk} = t_{a_j k}$$

for $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, m_j = m_{a_j}$.

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$$X \setminus V_j = \{Q_j = 0\}, \quad j = 1, 2, \ldots, m$$

for some regular functions Q_j on X.

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Then $s_{jk} = \psi_{i(j)}(t_{jk})$ and the sections s_{jk}, $k = 1, 2, \ldots, m_j$ generate $\mathcal{F}$ over V_j. The sets $X \setminus V_j$ are Zariski closed and thus are of the form

$$X \setminus V_j = \{Q_j = 0\}, \quad j = 1, 2, \ldots, m$$

for some regular functions Q_j on X. It follows from Lemma 9 that there exist a multi-blowup $\sigma : \widetilde{X} \to X$ and a positive integer N such that for each $j = 1, 2, \ldots, m$ the sections

$$(Q_j^N)^\sigma \sigma^* s_{jk} \quad k = 1, 2, \ldots, m_j,$$

extends to global sections $\widetilde{s_{jk}} \in \sigma^* \mathcal{F}(\widetilde{X})$. Since $\{\sigma^{-1}(V_j)\}_{j=1}^m$ is a Zariski open covering of $\widetilde{X}$, it is easy to check that the global sections

$$\widetilde{s_{jk}}, \quad j = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, m_j$$

generate the pull-back $(\sigma^* \mathcal{F})_y$ for every $y \in \widetilde{X}$. This finishes the proof.

Corollary 1

Let $\mathcal{F}$ be a coherent sheaf on a non-singular real affine variety X. Then there exists a multi-blowup $\sigma : \widetilde{X} \to X$ such that the pull-back $\sigma^*\mathcal{F}$ admits a global presentation:

$$\mathcal{O}^p_{\widetilde{X}} \to \mathcal{O}^q_{\widetilde{X}} \to \sigma^*\mathcal{F} \to 0$$

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Connections with works of Tognoli:

Corollary 2

Let $\mathcal{F}$ be a coherent sheaf on a non-singular real affine variety X. Then there exists a multi-blowup $\sigma : \widetilde{X} \to X$ such that $\sigma^*\mathcal{F}$ is an A-coherent sheaf.

Back to our example

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Example 2

Let $P = X^2(X-1)^2 + Y^2 \in \mathbb{R}[X,Y]$*. The polynomial* P *is irreducible and has only two zeros* $c_1 = (0,0)$ *and* $c_2 = (1,0)$ *in* $\mathbb{R}^2$*. Put* $U_i = \mathbb{R}^2 \setminus \{c_i\}$*. The transition function*

$$g_{2,1} : U_1 \cap U_2 \to GL(1,\mathbb{R}) = \mathbb{R}^*$$

$$(x,y) \mapsto P(x,y)$$

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defines a vector bundle of rank 1 over $\mathbb{R}^2$*. Global sections can be described as a pair* (s_1, s_2) *where* $s_i : U_i \to \mathbb{R}$ *are regular functions and* $g_{2,1}s_1 = s_2$*. Set* $s_i = \frac{f_i}{h_i}$ *where* f_i, h_i *are relatively prime polynomials. Then* $Pf_1h_2 = f_2h_1$*. Since* P *does not divide* h_2 *we obtain that* $f_2 = \lambda P f_1$ *and* $h_1 = \lambda^{-1}h_2$*, where* $\lambda \in \mathbb{R}^*$*. This shows that every algebraic global section of this bundle vanishes at* c_1*.*

Let $\sigma : X \to \mathbb{R}^2$ be the blowup of $\mathbb{R}^2$ at the origin,

$$X = \{(x, y, u : v) \in \mathbb{R}^2 \times \mathbb{P}^1(\mathbb{R}) : xv = uy\}.$$

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$$\sigma(x, y, u : v) = (x, y)$$

and let $\sigma^*\mathcal{F}$ be the pullback of $\mathcal{F}$. We can cover X with two open charts

$$\Omega_1 = \{(x, y, u : v) \in X : u \neq 0\} \text{ and } \Omega_2 = \{(x, y, u : v) \in X : v \neq 0\},$$

with local coordinates

$$(x, \frac{v}{u}) \text{ and } (\frac{u}{v}, y)$$

on Ω_1 and Ω_2 respectively.

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on Ω_1 and Ω_2 respectively. In these local coordinates σ is expressed by the formulas

$$\sigma(r, s) = (r, rs) \text{ and } \sigma(r, s) = (rs, s),$$

respectively. Obviously $c_2 \in \sigma(\Omega_1)$ and $c_2 \notin \sigma(\Omega_2)$.

A crucial observation is that the function

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is regular on X. Indeed,

$$F(r,s) = \frac{r^2((r-1)^2+s^2)}{r^2(1+s^2)} = \frac{(r-1)^2+s^2}{1+s^2}$$

on the first chart, and

$$F(r,s) = \frac{s^2(r^2(rs-1)^2+1)}{s^2(r^2+1)} = \frac{r^2(rs-1)^2+1}{r^2+1}$$

on the other chart. Moreover, the set

$$\{c \in X : F(c) = 0\} = \sigma^{-1}(c_2) \in \Omega_1$$

is a singleton.

Consequently, the pair $\widetilde{s} = (\widetilde{s_1}, \widetilde{s_2})$ with

$$\widetilde{s_1} = \frac{1}{(x^2+y^2)^\sigma} \text{ on } U_1^\sigma, \text{ and } \widetilde{s_2} = F \text{ on } U_2^\sigma$$

is a nowhere vanishing global section of a line bundle $\sigma^*\xi$ and of the locally free sheaf $\sigma^*\mathcal{F}$ of rank 1. Therefore, $\widetilde{s}$ generates the sheaf $\sigma^*\mathcal{F}$ as desired.