A BOHR-JESSEN TYPE THEOREM FOR THE EPSTEIN ZETA-FUNCTION

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$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s}=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}, \quad \sigma>1.$$

$\mathbb{C}$; $\operatorname*{Res}_{s=1}\zeta(s)=1$.

- H. Bohr, 1914.
- H. Bohr, B. Jessen, 1930.

Theorem 1 (Bohr, Jessen; *Acta Math.*, 1930)

Suppose that $\sigma>1$ is fixed. Then, for every closed rectangle R with edges parallel to the axes, the limit

$$\lim_{T\to\infty}\frac{1}{T}\mathrm{J}\left\{t\in[0,T]:\log\zeta(\sigma+it)\in R\right\}$$

exists, and depends only on σ and R.

$\mathrm{J}A$ – Jordan measure of a measurable set $A\subset\mathbb{R}$.

- H. Bohr, B. Jessen, 1932.

$$G = \left\{ s \in \mathbb{C} : \sigma > \frac{1}{2} \right\} \setminus \bigcup_{s_j = \sigma_j + it_j} \left\{ s = \sigma + it_j : \frac{1}{2} < \sigma \leq \sigma_j \right\},$$

s_j runs over all zeros of $\zeta(s)$ in the region $\left\{ s \in \mathbb{C} : \frac{1}{2} < \sigma < 1 \right\}$.

Theorem 2 (Bohr, Jessen; *Acta Math.*, 1932)

Suppose that $\sigma > \frac{1}{2}$ is fixed. Then, for every closed rectangle R with edges parallel to the axes, the limit

$$\lim_{T \to \infty} \frac{1}{T} \mathrm{J} \left\{ t \in [0, T] : \sigma + it \in G, \log \zeta(\sigma + it) \in R \right\}$$

exists, and depends only on σ and R.

- B. Jessen, A. Wintner, 1935.
- V. Borchsenius, B. Jessen, 1948.

- $\mathcal{B}(X)$ – the Borel σ-field of the space X.
- P_n, $n \in \mathbb{N}$, and P – probability measures on $(X, \mathcal{B}(X))$.
- $P_n \Rightarrow P$ as $n \to \infty$ if, for every real bounded continuous

function f on X,

$$\lim_{n\to\infty} \int_X f \,\mathrm{d}P_n = \int_X f \,\mathrm{d}P.$$

Theorem 3 (A. Laurinčikas; 1996)

Suppose that $\sigma > \frac{1}{2}$ is fixed. Then, on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure P_σ such that

$$\frac{1}{T}\mathrm{meas}\left\{t \in [0, T] : \zeta(\sigma + it) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$.

$\mathrm{meas}A$ – the Lebesgue measure of a measurable set A.

- $\Omega = \prod_p \gamma_p, \quad \gamma_p = \gamma = \{s \in \mathbb{C} : |s| = 1\}, \quad \forall p.$
- $(\Omega, \mathcal{B}(\Omega))\ \exists m_H \to (\Omega, \mathcal{B}(\Omega), m_H).$
- $\omega(p)$ – p^{th} component of an element $\omega \in \Omega$.
- $(\Omega, \mathcal{B}(\Omega), m_H)$,

$$\zeta(\sigma, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^\sigma}\right)^{-1}, \quad \sigma > \frac{1}{2}.$$

- $P_{\zeta,\sigma}(A) = m_H \{\omega \in \Omega : \zeta(\sigma, \omega) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}).$

Theorem 4 (Bagchi, 1981; Laurinčikas, 1996; Steuding, 2007)

Suppose that $\sigma > \frac{1}{2}$ is fixed. Then,

$$\frac{1}{T}\text{meas}\{t \in [0, T] : \zeta(\sigma + it) \in A\}, \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{\zeta,\sigma}$ as $T \to \infty$.

AIM

To obtain a limit theorem for the Epstein zeta-function $\zeta(s; Q)$.

- P. Epstein, 1930.
- T. Nakamura, Ł. Pańkowski, 2013.

- Q – a positive definite quadratic $n \times n$ matrix.
- $Q[\mathbf{x}] = \mathbf{x}^{\mathrm{T}} Q \mathbf{x} \quad (\mathbf{x} \in \mathbb{Z}^n)$.

The Epstein zeta-function

$$\zeta(s; Q) = \sum_{\mathbf{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}} (Q[\mathbf{x}])^{-s}, \quad \sigma > \frac{n}{2}.$$

$$\operatorname*{Res}_{s=\frac{n}{2}} \zeta(s; Q) = \frac{\pi^{n/2}}{\Gamma(n/2)\sqrt{\det Q}},$$

- $\pi^{-s}\Gamma(s)\zeta(s; Q) = (\det Q)^{-1/2} \pi^{s-n/2} \Gamma\left(\frac{n}{2} - s\right) \zeta\left(\frac{n}{2} - s; Q^{-1}\right)$.

$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$,

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}$$ – full modular group,

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),\ c \equiv 0 (\mathrm{mod}\, q) \right\}$$ – Hecke subgroup.

- *A holomorphic function F on $\mathbb{H}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ satisfying*

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^{\kappa} F(z), \quad \kappa \in \mathbb{N},$$

and holomorphic at cusps of $\Gamma_0(q)$ is called the modular form of weight κ and level q.

- *If F vanishes at all cusps, then it is called a cusp form.*
- *Let $\kappa > 2$,*

$$E_{\kappa}(z) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(q)} (cz+d)^{-\kappa}$$

is called an Eisenstein series.

- Suppose that $Q[\mathbf{x}] \in \mathbb{Z}$, $\forall \mathbf{x} \in \mathbb{Z}^n \setminus \{\underline{0}\}$, $\quad (Q[\mathbf{x}] = \mathbf{x}^{\mathrm{T}} Q\mathbf{x})$.
- $r_Q(m) = \#\{\mathbf{x} \in \mathbb{Z}^n : \ Q[\mathbf{x}] = m, \ m \in \mathbb{N}_0\}$,

$$\zeta(s; Q) = \sum_{m=1}^{\infty} \frac{r_Q(m)}{m^s}, \quad \sigma > \frac{n}{2}.$$

- O. M. Fomenko, 2002

$$\theta(z; Q) = \sum_{m=0}^{\infty} r_Q(m) \mathrm{e}^{2\pi i m z} \text{ – modular form of weight } \frac{n}{2};$$

$$\theta(z; Q) = E_Q(z) + F_Q(z):$$

$E_Q(z) = \sum_{m=0}^{\infty} e_Q(m)\mathrm{e}^{2\pi i m z}$ – Eisenstein series,
$F_Q(z) = \sum_{m=1}^{\infty} f_Q(m)\mathrm{e}^{2\pi i m z}$ – cusp form.

$$\zeta(s; Q) = \zeta(s; E_Q) + \zeta(s; F_Q), \qquad (1)$$

$$\zeta(s; E_Q) = \sum_{m=1}^{\infty} \frac{e_Q(m)}{m^s}, \quad \zeta(s; F_Q) = \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}, \quad \sigma > \frac{n}{2}.$$

- H. Iwaniec, 1997

If n – even and $n \geq 4 \Longrightarrow E_Q(z)$ – a modular form of weight $\frac{n}{2}$ and level q (where $q \in \mathbb{N}$ is such that $q(2Q)^{-1}$ is an integral matrix).

- E. Hecke, 1937

$\zeta(s; E_Q)$ – a certain linear combination of elements

$$(kl)^{-s} L(s, \chi_k) L\left(s - \frac{n}{2} + 1, \varphi_l\right),$$

k and l are positive divisors of q;
χ_k and φ_l – Dirichlet characters modulo $\frac{q}{k}$ and $\frac{q}{l}$, respectively.

————————- $\Downarrow$ + (1): $\zeta(s; Q) = \zeta(s; E_Q) + \zeta(s; F_Q)$ ————————-

$$\zeta(s; Q) = \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}}{k^s l^s} L(s, \chi_k) L\left(s - \frac{n}{2} + 1, \varphi_l\right) + \sum_{m=1}^{\infty} \frac{f_Q(m)}{m^s}, \ \sigma > \frac{n-1}{2}.$$

- $f_Q(m) = O\left(m^{n/4 - 1/2 + \varepsilon}\right), \ \varepsilon > 0, \quad \Longrightarrow$

$$\sum^{\infty} \frac{f_Q(m)}{m^s} \text{ converges absolutely for } \sigma > \frac{n-1}{2}.$$

- $\gamma = \{s \in \mathbb{C} : |s| = 1\}$,

$$\Omega = \prod_p \gamma_p, \quad \gamma_p = \gamma, \quad \forall p.$$

- Extend $\omega(p)$ to the set $\mathbb{N}$:

$$\omega(m) = \prod_{\substack{p^\alpha | m \\ p^{\alpha+1} \nmid m}} \omega^\alpha(p), \quad m \in \mathbb{N},$$

- $\sigma > \frac{n-1}{2}$, $(\Omega, \mathcal{B}(\Omega), m_H)$,

$$\begin{aligned} \zeta(\sigma, \omega; Q) &= \sum_{k=1}^{K} \sum_{l=1}^{L} \frac{a_{kl}\omega(k)\omega(l)}{k^\sigma l^\sigma} L(\sigma, \omega, \chi_k) L\left(\sigma - \frac{n}{2} + 1, \omega, \varphi_l\right) \\ &+ \sum_{m=1}^{\infty} \frac{f_Q(m)\omega(m)}{m^\sigma}, \end{aligned}$$

$$L(\sigma, \omega, \chi_k) = \prod_p \left(1 - \frac{\chi_k(p)\omega(p)}{p^\sigma}\right)^{-1}, \; L(\sigma - \frac{n}{2} + 1, \omega, \varphi_l) = \prod_p \left(1 - \frac{\varphi_l(p)\omega(p)}{p^{\sigma - n/2 + 1}}\right)^{-1}.$$

$$P_{\zeta,\sigma;Q}(A) = m_H\left\{\omega \in \Omega : \zeta(\sigma,\omega;Q) \in A\right\}, \quad A \in \mathcal{B}(\mathbb{C}).$$

Theorem (A. Laurinčikas, R. Macaitienė, 2018 (not published yet))

Suppose that $\sigma > \frac{n-1}{2}$ is fixed. Then

$$\frac{1}{T}\mathrm{meas}\left\{t \in [0,T] : \zeta(\sigma+it;Q) \in A\right\}, A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to the measure $P_{\zeta,\sigma;Q}$ as $T \to \infty$.

THANK YOU
FOR YOUR ATTENTION!

DZIĘKUJĘ ZA UWAGĘ!