Graded quaternion-symbol equivalence

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ALaNT 5

Fundamental question

To what extend the arithmetic of a field determines possible geometries over it?

- Take $V = K^3$ equipped with a quadratic form $x^2 + y^2 + z^2$ (normal dot-product).
- Does it contain a self-orthogonal (isotropic) vector?
- For $K = \mathbb{Q}(\sqrt{5})$: NO
- For $K = \mathbb{Q}(\sqrt{-5})$: YES

- So, geometry may depend on arithmetic!

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- So, geometry may depend on arithmetic!

Philosophical question

To what extend does geometry depends on arithmetic?

Mathematical question

- $\mathcal{F}$ category of fields,
- $\mathcal{R}$ category of commutative rings,
- $W : \mathcal{F} \to \mathcal{R}$ Witt functor.

When $WK \cong WL$ for two fields K, L?

- 1970 D.K. Harrison: general criterion using isomorphism of square class groups,
- 1973–85 A.B. Carson, C. Cordes, M. Kula, M. Marshall, L. Szczepanik, K. Szymiczek: fields with ≤ 32 squares classes,
- 1990s P.E. Conner, A. Czogała, R. Litherland, R. Perlis, K. Szymiczek: global fields,
- 2002 K.: real function fields
- 2013 N. Grenier-Boley, D.W. Hoffmann: real SAP fields with (general) u-invariant ≤ 2
- 2017 P. Gładki, M. Marshall: function fields over local and global fields

Given a field K denote:

- $\mathrm{Br}(K)$ the Brauer group of similarity classes of central simple algebras,
- $\mathrm{BW}(K)$ the Brauer-Wall group o similarity classes of central simple graded algebras,
- $\mathrm{Q}(K)$ the subgroup of $\mathrm{Br}(K)$ generated by classes of quaternion algebras,
  Merkurjev (1981): $\mathrm{Q}(K) = \{A \in \mathrm{Br}(K) \mid A^2 = 1\}$.
- $\mathrm{GQ}(K)$ the subgroup of $\mathrm{Br}(K)$ generated by classes of graded quaternion algebras.

Let:

- K, L be two fields,
- Ω_K, Ω_L certain sets of places/valuations on K, L,
- $t\colon {}^{K}/\Box \xrightarrow{\sim} {}^{L}/\Box$ is an isomorphism,
- $T\colon \Omega_K \xrightarrow{\sim} \Omega_L$ is a bijection.

The pair (t, T) is a *quaternion-symbol equivalence*
(a.k.a: reciprocity equivalence, Hilbert-symbol equivalence), if

$$\Gamma_v\colon \mathrm{Q}(K_v) \to \mathrm{Q}(L_{Tv}), \quad \Gamma_v\left(\frac{a, b}{K_v}\right) := \left(\frac{ta, tb}{L_{Tv}}\right)$$

induces a group homomorphism for every $v \in \Omega_K$

Theorem (Perlis, Szymiczek, Conner, Litherland)

Assume

- K, L *global fields,*
- $\operatorname{char} K, \operatorname{char} L \neq 2$,
- Ω_K, Ω_L *all places of* K, L

Then the following conditions are equivalent:

- $WK \cong WL$,
- *there is a quaternion-symbol equivalence.*

Consequences of the previous theorem:

SZYMICZEK, 1991:

Complete set of invariants for Witt equivalence.

CZOGAŁA, K., 2018

Algorithm for testing Witt equivalence of algebraic number fields.

Theorem (K., 2002)

Assume

- $\mathbb{k}$ *fixed real closed field,*
- K, L *real algebraic function fields over* $\mathbb{k}$*,*
- Ω_K, Ω_L *almost all real places of* K, L *trivial on* $\mathbb{k}$*.*

Then the following conditions are equivalent:

- $WK \cong WL$*,*
- *there is a quaternion-symbol equivalence.*

In this case:

- T is a homeomorphism of the associated real curves (except finitely many points),
- every such a homeomorphism gives raise to a quaternion-symbol equivalence and consequently to a Witt equivalence.

Corollary (K. 2002 / Grenier-Boley–Hoffmann 2013)

Every two formally real function fields over a fixed real closed field are Witt equivalent.

Theorem (Gładki–Marshall, 2017)

Assume:

- k, l *are global fields,*
- K, L *are function fields over* k, l*,*
- Ω_K, Ω_L *are sets of all nontrivial Abhyankar valuations s.t. the residue field are infinite and* $\mathrm{char} \neq 2$*.*

Then Witt equivalence implies quaternion-symbol equivalence.

Let's alter the definition a bit?

(Original motivation/hope was to get a finer classification of fields.)

Let:

- K, L be two fields,
- Ω_K, Ω_L certain sets of places/valuations on K, L,
- $t\colon {}^{K}\!/\square \xrightarrow{\sim} {}^{L}\!/\square$ is an isomorphism,
- $T\colon \Omega_K \xrightarrow{\sim} \Omega_L$ is a bijection.

The pair (t, T) is a *graded* quaternion-symbol equivalence, if

$$\left\langle \frac{a, b}{K_v} \right\rangle \mapsto \left\langle \frac{ta, tb}{L_{Tv}} \right\rangle$$

induces a group isomorphism $\Lambda_v\colon \operatorname{GQ}(K_v) \xrightarrow{\sim} \operatorname{GQ}(L_{Tv})$ for every $v \in \Omega_K$.

On one hand:

- $\mathrm{GQ}(K_v)$ is in general "bigger" than $\mathrm{Q}(K_v)$,
- hence an isomorphism gives a "finer-grain control";

On the other hand:

- $\left(\frac{a,b}{K_v}\right) = 1$ iff $\langle 1, a\rangle \otimes \langle 1, b\rangle$ is hyperbolic over K_v,
- hence in QSE, we "control" 2-fold Pfister forms
- $\left\langle\frac{a,b}{K_v}\right\rangle = 1$ iff $\langle a, b\rangle$ is hyperbolic over K_v;
- hence, we "control" only binary forms;
- thus, GQSE might be a weaker condition.

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OBSERVATION

In general

$$\textit{graded equivalence} \;\not\Leftrightarrow\; \textit{"ungraded" equivalence}$$

- $K = L = \mathbb{R}(x)((y))$,
- $\Omega_K = \Omega_L = \{$ the unique valuation trivial on $\mathbb{R}(x)\}$,
- T identity
- $\mathcal{B}$ a $\mathbb{F}_2$-basis of ${}^{K}/\square$ containing $\{-1, x, x^2+1\}$
- t defined on basis $\mathcal{B}$ as follows:

$$t(x) = x^2 + 1, \qquad t(x^2+1) = x$$
$$t(v) = v \quad \text{for} \quad v \in \mathcal{B} \setminus \{x, x^2+1\}$$

Then

- (t, T) is a graded quaternion-symbol equivalence
- (t, T) is not a quaternion-symbol equivalence

Question

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Observation

There is a canonical bijection

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Observation

There is a canonical bijection

$$\mathrm{GQ}(K_v) \cong \mathrm{Q}(K_v) \times {}^{K_v}/\square.$$

In general it is not a group isomorphism!

"there's the rub"
(W. Shakespeare)

Can we do better if we restrict ourselves to specific classes of fields?
Global fields? Real function fields?

Assume:

- K a global field,
- Ω_K set of all places of K.

Then for $v \in \Omega_K$:

1. if $K_v \cong \mathbb{C}$, then $|\operatorname{GQ}(K_v)| = 1$;
2. if $K_v \cong \mathbb{R}$, then $|\operatorname{GQ}(K_v)| = 4$ and $\operatorname{GQ}(K_v) \cong \mathbb{Z}_4$;
3. if K_v is local non-dyadic, then $|\operatorname{GQ}(K_v)| = 8$ and
$$-1 \in K_v^{\times 2} \Longrightarrow \operatorname{GQ}(K_v) \cong \mathbb{Z}_2^3$$
$$-1 \notin K_v^{\times 2} \Longrightarrow \operatorname{GQ}(K_v) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$
4. if K_v is local dyadic, then $|\operatorname{GQ}(K_v)| = 2^{n+3}$, where $n = (K_v : \mathbb{Q}_2)$.

Lemma

If $K_v = \mathbb{R}$ or K_v is a local field, then $\mathrm{GQ}(K_v)$ is a disjoint sum

$$\mathrm{GQ}(K_v) = \left\{ \left\langle \tfrac{a,b}{K_v} \right\rangle \colon a, b \in {}^{K_v}\!/\square \right\} \cup \left\{ A_v \right\},$$

where A_v is an explicitly given distinguished element.

LEMMA

If $K_v = \mathbb{R}$ or K_v is a local field, then $\mathrm{GQ}(K_v)$ is a disjoint sum

$$\mathrm{GQ}(K_v) = \left\{ \left\langle \tfrac{a,b}{K_v} \right\rangle : a, b \in {}^{K_v}/\square \right\} \cup \left\{ A_v \right\},$$

where A_v is an explicitly given distinguished element.

Moreover, this element is preserved by every graded quaternion-symbol equivalence!

COROLLARY

A graded quaternion-symbol equivalence of global fields preserves:

- *complex places,*
- *real places,*
- *finite non-dyadic places,*
- *dyadic places and local dyadic degrees,*
- *local squares and local minus squares,*
- *local levels,*
- -1,
- *global level.*

COROLLARY[2]

Let K, L be global fields.

- If there is a graded quaternion-symbol equivalence,
- then there is a quaternion-symbol equivalence between K and L.

Theorem

Let K, L be global fields and assume

- *K has no more than one dyadic place.*

Then every graded quaternion-symbol equivalence (t, T) is a quaternion-symbol equivalence.

Examples: global function fields,
global number fields where 2 does not split at all.

Proposition

If K, L are global fields, then every quaternion-symbol equivalence (t, T) is a graded quaternion-symbol equivalence.

Theorem

Let K, L be global fields. The following conditions are equivalent

- $WK \cong WL$;
- *there is a quaternion-symbol equivalence between K and L;*
- *there is a graded quaternion-symbol equivalence.*

How about real function fields?

Proposition

Assume

- $\mathbb{k}$ *is a real closed field,*
- K, L *are real function fields,*
- Ω_K, Ω_L *are all the real places of* K, L *trivial on* $\mathbb{k}$*.*

Then

- *every graded quaternion-symbol equivalence is a quaternion-symbol equivalence;*
- *every quaternion-symbol equivalence is a graded quaternion-symbol equivalence.*

Thank you for your attention.