Annihilators of the minus class group of an imaginary cyclic field

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Our goal is to find annihilators of A_ℓ^-, i.e. the elements $\xi \in \mathbb{Z}[G]$ (or $\mathbb{Z}_\ell[G]$) such that for any ideal $\mathfrak{a}$ representing arbitrary class of A_ℓ^- the ideal $\mathfrak{a}^\xi$ is principal.

Kummer, Stickelberger, Sinnott

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For any $a \in \mathbb{Z}[\zeta_n]$ such that $\mathfrak{p} \nmid a$ we define (n-th power residue symbol)

$$\omega(a \bmod \mathfrak{p}) \equiv a^{(p-1)/n} \pmod{\mathfrak{p}}.$$

Using this character we form a Gauss sum

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$$\theta_n = \sum_{\substack{1 \le a < n \\ (a,n)=1}} \frac{a}{n} \cdot \sigma_a^{-1} \in \mathbb{Q}[\mathrm{Gal}\,(\mathbb{Q}_n/\mathbb{Q})],$$

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which does not depend on p! Since each ideal class contains such an ideal $\mathfrak{p}$, it follows that

$$(\theta_n \cdot \mathbb{Z}[\mathrm{Gal}\,(\mathbb{Q}_n/\mathbb{Q})]) \cap \mathbb{Z}[\mathrm{Gal}\,(\mathbb{Q}_n/\mathbb{Q})]$$

annihilates the ideal class group of $\mathbb{Q}_n$.

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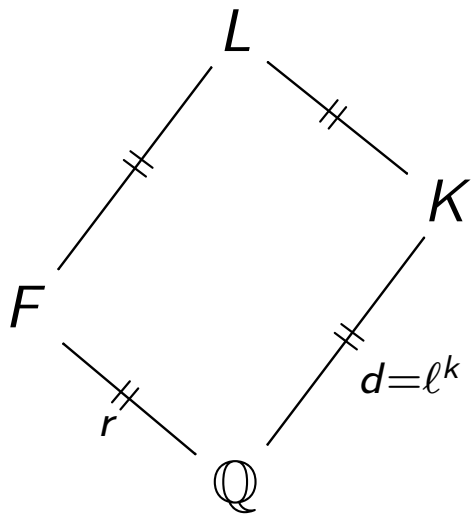

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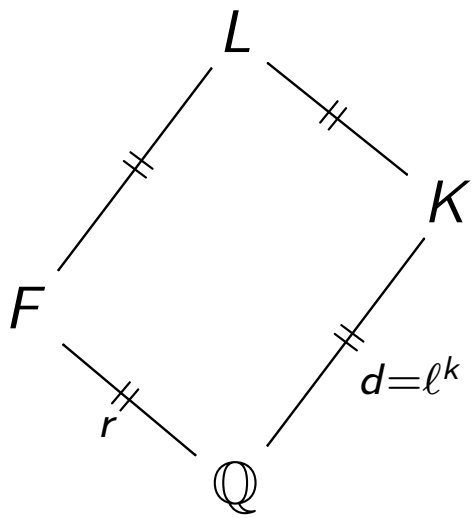

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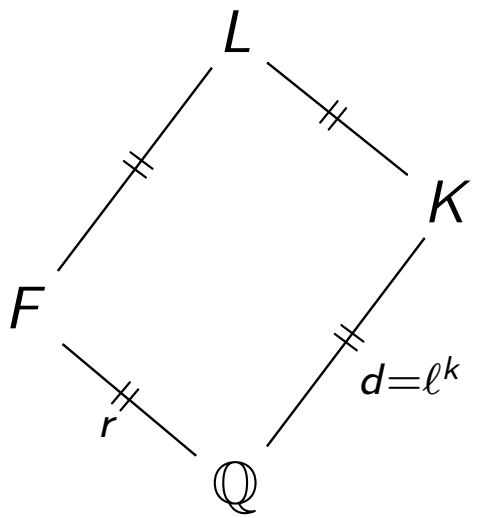

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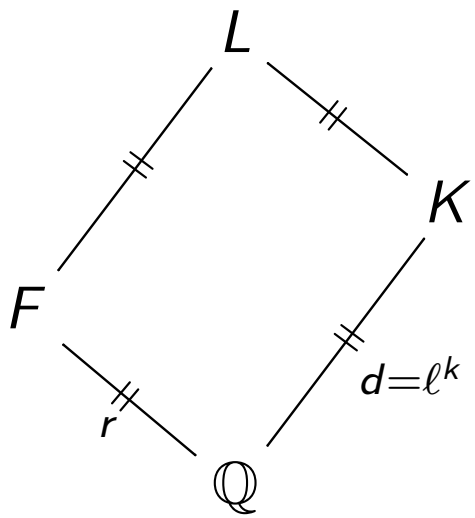

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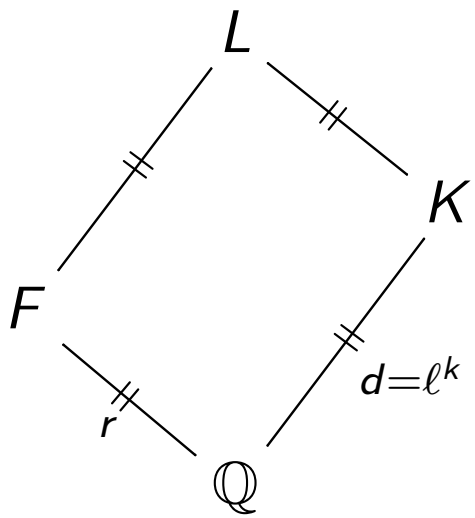

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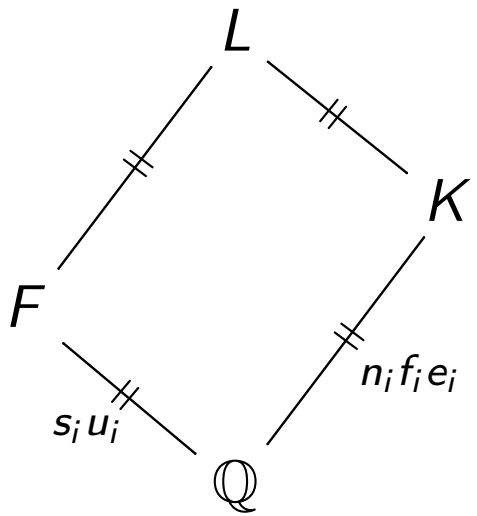

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Proposition

Let $\delta = H(\gamma)$, where

$$H(X) = \frac{\prod_{i=1}^{t+1} h_i(X)}{h(X)}.$$

Then there is $\beta \in L$ such that $\mathrm{N}_{\mathbb{Q}_{fm}/L}\left(g^{fm(1-\tau)}\right)^{2f} = \beta^{\delta}$.

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$$\varkappa_L = 2f^2 m \cdot (1 - \gamma^{r\ell^k/2}) \cdot \mathrm{res}_{\mathbb{Q}_{fm}/L}\theta_{fm} \in \mathbb{Z}[\langle\gamma\rangle].$$

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Theorem

The element $\xi_L \in \mathbb{Z}[\langle\gamma\rangle]$ is an annihilator of A_ℓ^-.

Let $\mathcal{L}$ be a poset consisting of the subfields M of L which are imaginary and satisfy $M = \mathbb{Q}_n \cap L$ for suitable positive integer n.

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Moreover $\mathrm{ord}_\ell\big([\mathcal{J} : \mathcal{I}]\big)$ divides the relative class number of L.

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Remark: If $s_k \geq 2$, then our new (explicitly computed) annihilators lie outside of the Stickelberger ideal S_L!

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Since both 13 and 37 split completely in F, we have

$$H(X) = \frac{(X^2-1)^2(X-1)}{(X^2-1)} = (X^2-1)(X-1) = \Phi_1(X)^2\Phi_2(X).$$

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Then we have

$$\begin{aligned}\xi_L &= \left(\frac{1}{6}\Delta_1\right)^2 \cdot \left(\frac{1}{3}\Delta_2(\gamma - 1)\right) \varkappa_L \\ &= 2^6 \cdot 13 \cdot 37 \cdot (-\gamma^5 - 2\gamma^4 - \gamma^3 + \gamma^2 + 2\gamma + 1).\end{aligned}$$

Thank you for your attention!