Intrinsic factorization of ideals in Dedekind domains

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Introduction

Introduction I

Let R be a Dedekind domain. Every ideal $\mathfrak{a} \lhd R$ has a unique factorization into a product of powers of prime ideals.

There are cases where this factorization is algorithmically computable. For instance, if $R = \mathbb{Z}_K$, $K = \mathbb{Q}(\vartheta)$ in [1] or [2].


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Advanced topics in computational number theory, volume 193 of *Graduate Texts in Mathematics*.


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The algorithms can be adapted also to global function fields. They depend however on knowing an embedding of the ring of integers (or polynomials) into R.

Introduction II

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The ideal to be factored passes through a three-stage process:

- Radical decomposition
- Distinct degree factorization
- Equal degree factorization

Radical decomposition of ideals

Radical decomposition I

Let R be a Dedekind domain and $\mathfrak{a} \lhd R$. Assume

$$\mathfrak{a} = \mathfrak{p}_1^{k_1} \cdots \mathfrak{p}_s^{k_s}, \tag{1}$$

For any $j \leq m := \max\{k_1, \ldots, k_s\}$ denote

$$\mathfrak{g}_j := \bigcap_{\substack{1 \leq i \leq s \\ k_i = j}} \mathfrak{p}_i.$$

This way we may write $\mathfrak{a}$ as a product

$$\mathfrak{a} = \mathfrak{g}_1 \cdot \mathfrak{g}_2^2 \cdots \mathfrak{g}_m^m. \tag{2}$$

and call 2 the radical decomposition of the ideal $\mathfrak{a}$

Radical decomposition II

The following operations are the basic building blocks for our first algorithm. Given $\mathfrak{a}, \mathfrak{b} \lhd R$, compute

- radical $\operatorname{rad}(\mathfrak{a})$
- sum $(\mathfrak{a} + \mathfrak{b})$
- colon $(\mathfrak{a} : \mathfrak{b})$

If all three operations are computable in R, then R is said to be a ring with computable ideal arithmetic.

Radical decomposition III

Proposition 2

Let $\Bbbk$ be a perfect, computable field and $C := \{F = 0\}$ be a smooth, geometrically irreducible algebraic curve over $\Bbbk$, defined by a bivariate polynomial $F \in \Bbbk[X, Y]$. Then the coordinate ring $R = \Bbbk[C] = {}^{\Bbbk[X,Y]}/_{\langle F \rangle}$ admits computable ideal arithmetic.

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Lemma 3

Keep the assumptions of the proposition. If $\mathfrak{a}, \mathfrak{b} \lhd R$ are two ideals, then

$$\mathfrak{a}^{ce} = \mathfrak{a} \qquad \operatorname{rad}(\mathfrak{a}) = \left(\operatorname{rad}(\mathfrak{a}^c)\right)^e \qquad (\mathfrak{a} : \mathfrak{b}) = (\mathfrak{a}^c : \mathfrak{b}^c)^e$$

Algorithm 1 – Radical Decomposition of an ideal

Given $\mathfrak{a} \lhd R$. Define three sequences $(\mathfrak{a}_i)$, $(\mathfrak{b}_i)$, $(\mathfrak{g}_i)$ as follows:

1. Let $\mathfrak{a}_0 \leftarrow \mathfrak{a}$, $\mathfrak{b}_1 \leftarrow \mathrm{rad}(\mathfrak{a})$ and $\mathfrak{a}_1 = (\mathfrak{a} : \mathfrak{b})$
2. Compute $\mathfrak{b}_{i+1} \leftarrow \mathfrak{a}_i + \mathfrak{b}_i$. The ideal $\mathfrak{b}_i$ satisfy:

$$\mathfrak{b}_i = \mathfrak{g}_i \cdot \mathfrak{g}_{i+1} \cdots \mathfrak{g}_m$$

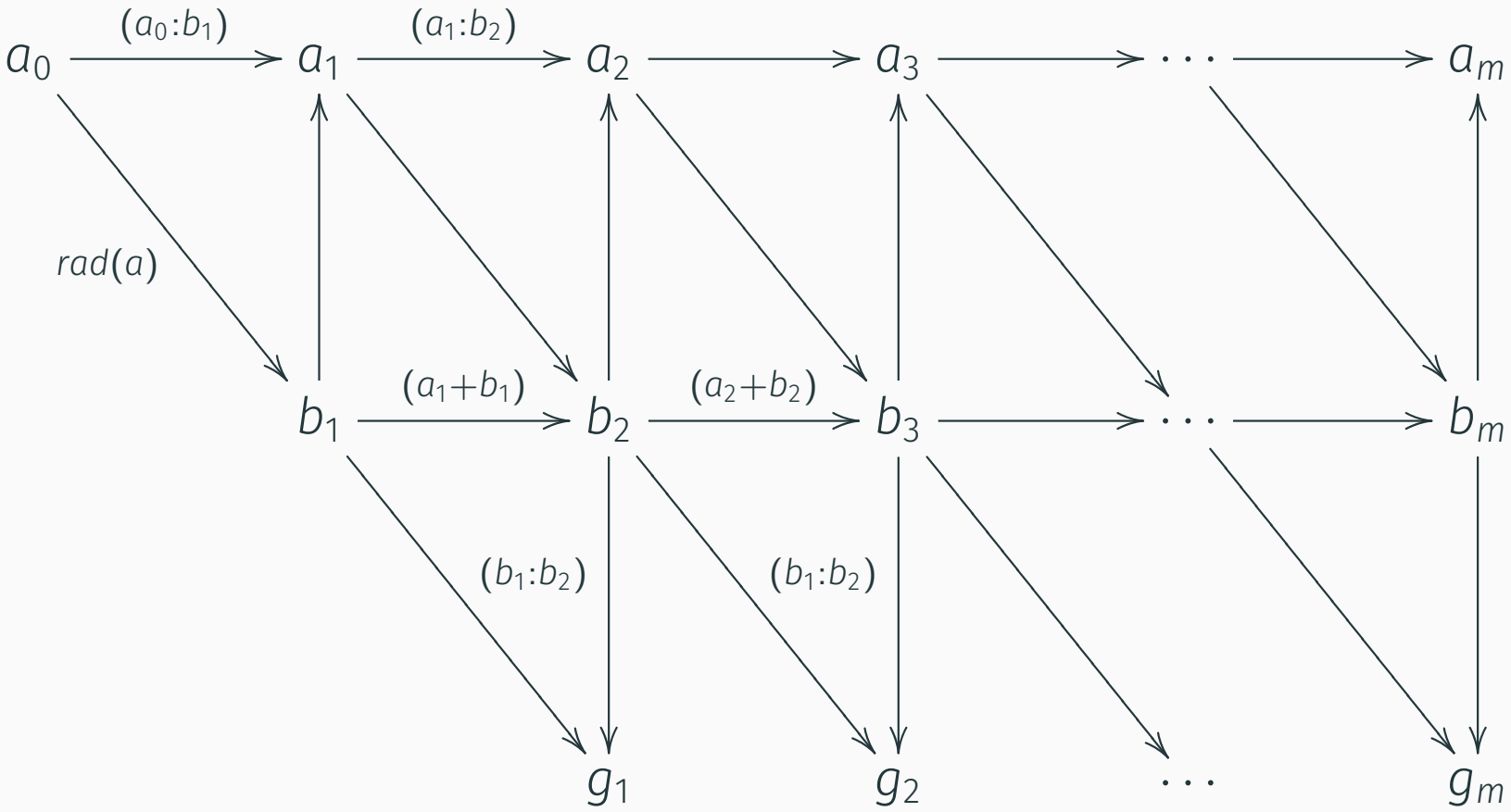
3. Compute $\mathfrak{a}_{i+1} \leftarrow (\mathfrak{a}_i : \mathfrak{b}_{i+1})$. The ideal $\mathfrak{a}_i$ satisfy:

$$\mathfrak{a}_i = \mathfrak{g}_{i+1} \cdot \mathfrak{g}_{i+2}^2 \cdots \mathfrak{g}_m^{m-i}$$

4. Finally we build the sequence $(\mathfrak{g}_1, \ldots, \mathfrak{g}_i)$ as

$$\mathfrak{g}_i = (\mathfrak{b}_i : \mathfrak{b}_{i+1})$$

Algorithm 1 – Diagram



Distinct degree factorization

Distinct degree factorization I

Let $\Bbbk$ be a fixed finite field and let $R = \Bbbk[C] = \Bbbk[X, Y]/\langle F \rangle$. Given a radical ideal $\mathfrak{a} \lhd R$, consider its factorization into primes

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_s.$$

Collate the primes with respect to their residual degrees setting

$$\mathfrak{h}_j := \prod_{\substack{\mathfrak{p} \mid \mathfrak{a} \\ \deg \mathfrak{p} = j}} \mathfrak{p}.$$

Consequently the ideal $\mathfrak{a}$ may be expressed as a product

$$\mathfrak{a} = \mathfrak{h}_1 \cdots \mathfrak{h}_m, \qquad \text{where} \quad m := \max\{\deg \mathfrak{p} \mid \mathfrak{p} \text{ divides } \mathfrak{a}\}. \tag{3}$$

We shall call (3) the distinct degree factorization of $\mathfrak{a}$.

Distinct degree factorization II

We will compute the distinct degree factorization of a given ideal $\mathfrak{a}$ by constructing successive greatest common divisors of $\mathfrak{a}$ and $\mathfrak{u}_k$,

$$\mathfrak{u}_k := \prod_{\substack{\mathfrak{p} \text{ prime} \\ \deg \mathfrak{p} | k}} \mathfrak{p}.$$

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Let $(x_\mathfrak{p}, y_\mathfrak{p})$ be the unique points associated with $\mathfrak{p}$ on the curve C, with coordinates in the algebraic closure of $\Bbbk = \mathbb{F}_q$, $q = p^l$.

The degree of $\mathfrak{p}$ divides k if and only if $x_\mathfrak{p}, y_\mathfrak{p}$ lie in $\mathbb{F}_{q^k}$ consists of elements satisfying $a^{q^k} - a = 0$.

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Lemma 4

For every $k \geq 1$, the ideal $\mathfrak{u}_k$ is generated by $x^{q^k} - x$ and $y^{q^k} - y$, where x, y are images of $X, Y \in \Bbbk[X, Y]$ in R .

Algorithm 1 – Radical Decomposition of an ideal

Given a radical $\mathfrak{a} \lhd R$ we compute the distinct degree factorization as follows.

1. Initialize $k \leftarrow 1$ and $\mathfrak{a}_1 \leftarrow \mathfrak{a}$
2. Compute $\mathfrak{u}_k \leftarrow \left\langle x^{q^k} - x, y^{q^k} - y \right\rangle$
3. Next compute $\mathfrak{h}_k \leftarrow \mathfrak{u}_k + \mathfrak{a}_k$
4. Compute $\mathfrak{a}_{k+1}$ from the colon $(\mathfrak{a}_k : \mathfrak{h}_k)$
5. Increase k by 1 and repeat until $\mathfrak{a}_k = R$.
6. Return $\mathfrak{h}_1, \ldots, \mathfrak{h}_k$

Equal degree factorization

Equal degree factorization I

- 👍 radical decomposition
- 👍 distinct degree factorization

we are left with a list of radical ideals

$$\mathfrak{h}_1, \ldots, \mathfrak{h}_k \quad \text{s.t.} \quad \mathfrak{h}_i := \prod_{\substack{\mathfrak{p} \text{ prime} \\ \deg \mathfrak{p} \mid k}} \mathfrak{p}$$

We can deal with such ideals using a generalization of a classical Cantor–Zassenhaus algorithm for factoring polynomials over finite fields.

Equal degree factorization II

Let $\mathfrak{a} \lhd R$ be a radical ideal with some unknown factorization

$$\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_m$$

and the residual degrees of $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are all the same and known. Denote this common degree by d.

Lemma 6

Let b be an element of R not in $\mathfrak{a}$. Denote $\overline{b} := b + \mathfrak{a}$ the class of b in $R/\mathfrak{a}$ and $e := q^d - 1$. The following conditions are equivalent:

1. the ideal $\mathfrak{b} := \langle b \rangle + \mathfrak{a}$ is a proper divisor of $\mathfrak{a}$;
2. the element $\overline{b}$ is a zero-divisor in $R/\mathfrak{a}$;
3. $\overline{b}^e \neq 1$.

Algorithm 3 – Equal degree factorization

Input: a radical ideal $\mathfrak{a} \lhd R$ and an integer d such that the residual degree of every prime factor of $\mathfrak{a}$ equals d.
Output: prime factors $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of $\mathfrak{a}$.

1. Select a random element $b \in R \setminus \mathfrak{a}$.
2. Set $\overline{b} \leftarrow b + \mathfrak{a} \in {}^{R}/_{a}$ the class of b in ${}^{R}/_{a}$.
3. Compute $\overline{b}^{e}$ which is either 1 if $b \notin \mathfrak{p}_i$, or 0 if $b \in \mathfrak{p}_i$.
4. If $\overline{b}^{q^d-1} \neq 1$, set $\mathfrak{b} \leftarrow \langle b \rangle + \mathfrak{a}$ and $\mathfrak{c} \leftarrow (\mathfrak{a} : \mathfrak{b})$.
5. $r_1 \leftarrow$ Equal degree factorization of $\mathfrak{b}$
6. $r_2 \leftarrow$ Equal degree factorization of $\mathfrak{c}$
7. Repeat until $|{}^{R}/_{\mathfrak{a}}| = q^d$ and return $r_1 \cup r_2$

Algorithm 4 – Complete factorization

Input: an ideal $\mathfrak{a}$ in R.
Output: the list of pairs $(\mathfrak{p}_i, k_i)$ of prime divisors and multiplicities, see Eq. (1).

1. $Factors = []$.
2. $G \leftarrow$ radical decomposition of $\mathfrak{a}$ (Algorithm 1).
3. Set $\mathfrak{g}_j \leftarrow G[j]$.
4. $H \leftarrow$ distinct degree factorization of $\mathfrak{g}_j$ (Algorithm 2).
5. Set $\mathfrak{h}_d \leftarrow H[d]$.
6. $P \leftarrow$ equal degree factorization of $\mathfrak{h}_d$ (Algorithm 3).
7. $Factors \leftarrow Factors \cup \big[(\mathfrak{p}, j) : \mathfrak{p} \in P\big]$

Example

Example

- $K = \mathbb{F}_{13}(x, y)$
- $F = y^2 - (x^5 - x)(x^4 + 2)$
- $R := \mathbb{F}_{13}[x, y]/\langle F \rangle$.

Consider the ideal $\mathfrak{a} \lhd R$

$$\mathfrak{a} = \langle x^9 + 8x^7 + 5x^6 + 10x^5 + 6x^4 + 4x^3 + 9x^2 + 6x + 4,$$
$$11x^8 + 8x^7 + 2x^6 + 10x^5 + 6x^4 + x^3y + x^3 + 4x^2y + 7x^2 + 4xy + 9y + 7\rangle$$

1. Compute the radical decomposition $\mathfrak{a} = \mathfrak{g}_1 \cdot \mathfrak{g}_2^2$ using Algorithm 1

$$\mathfrak{g}_1 = \langle x^6 + 9x^5 + 7x^4 + 10x^3 + 4x^2 + 4x + 12,$$
$$y + 12x^5 + x^4 + 11x^3 + 10x^2 + 3x + 8\rangle$$
$$\mathfrak{g}_2 = \langle x^3 + 4x^2 + 4x + 9, y + 7x^2 + 9x + 12\rangle$$

Example

2. Next, using Algorithm 2, we compute the distinct degree factorization for each element of the radical decomposition.
   - For $\mathfrak{g}_1$ it returns $\mathfrak{h}_{11} = \mathfrak{h}_{12} = R$ and

$$\mathfrak{h}_{13} = \langle 8x^5y + 5x^4y + 9x^3y + xy + 5y + 1, \\ x^6y + 9x^5y + 7x^4y + 10x^3y + 4x^2y + 4xy + 12y \rangle.$$

   - For $\mathfrak{g}_2$ it returns $\mathfrak{h}_{21} = \mathfrak{h}_{22} = R$ and

$$\mathfrak{h}_{23} = \left\langle 5x^2y + 5xy + 6y + 1, x^3y + 4x^2y + 4xy + 9y \right\rangle.$$

Example

3. Finally we compute the equal degree factorization for each of the above factors using Algorithm 3.
   - For $\mathfrak{h}_{13}$ we obtain the following primes

$$\mathfrak{p}_1 = \left\langle x^3 + 4x^2 + 4x + 9, y + 6x^2 + 4x + 1 \right\rangle$$
$$\mathfrak{p}_2 = \left\langle x^3 + 5x^2 + 9x + 10, y + 3x^2 + 7x + 4 \right\rangle$$

   - and for $\mathfrak{h}_{23}$ we get

$$\mathfrak{p}_3 = \left\langle x^3 + 4x^2 + 4x + 9, y + 7x^2 + 9x + 12 \right\rangle$$

Hence the complete factorization of $\mathfrak{a}$ is $\mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3^2$.

Thank You!

Any Questions?