Prüfer domains of integer-valued polynomials over subsets

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Prüfer domains

Definition

An integral domain R is a *Prüfer domain* if for each maximal ideal M of R, the localization R_M is a valuation domain.

A Prüfer domain is integrally closed, since it is equal to the intersection of its valuation overrings V, i.e., $R \subset V \subset QF(R) = K$.

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Question: given a family of valuation domains V_i, $i \in I$, of a field K, when is their intersection $\bigcap_{i \in I} V_i$ a Prüfer domain (with quotient field K)?

- if I is finite, then $\bigcap_{i \in I} V_i$ is Prüfer (Nagata's Theorem).

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Example

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

is a non-Noetherian Prüfer domain of Krull dimension 2.

Rings of integer-valued polynomials over subsets

Definition

Let D be an integral domain with quotient field K. For each $S \subseteq D$ we define the *ring of integer-valued polynomials over S* as:

$$\text{Int}(S, D) = \{f \in K[X] \mid f(S) \subseteq D\}$$

For $S = D$, we set $\text{Int}(D, D) = \text{Int}(D)$.

Question: for which subset S of D is $\text{Int}(S, D)$ a Prüfer domain?

For $S = D$ Loper gave a complete characterization in 1998. For general subsets, the question is more subtle...

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A necessary condition

If $\mathrm{Int}(S, D)$ is Prüfer, then D is Prüfer.

From now on, we work locally: $D = V$ a valuation domain.

Known results

Theorem (Chabert 1987)

Int(V) is Prüfer if and only if V is a DVR with finite residue field.

Consequence: since for $S \subseteq V$ we have $\text{Int}(V) \subseteq \text{Int}(S, V)$, whenever V is a DVR with finite residue field, $\text{Int}(S, V)$ is automatically Prüfer.

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Recall that a subset S of V is *precompact* if the topological closure $\widehat{S}$ of S in the completion $\widehat{V}$ is compact.

Theorem (Cahen, Chabert, Loper, 2000)

Let $S \subseteq V$ be a precompact subset. Then $\text{Int}(S, V)$ is Prüfer.

Remark

The precompactness is a necessary condition when V is a DVR (CCL, 2000) or when S is an additive subset of V (Park, 2016).

Precompactness is not necessary (after Loper and Werner)

Definition (Ostrowski)

Let $V \subset K$ be a valuation domain with associated valuation v.
A sequence $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ is said to be *pseudo-convergent* if $v(s_n - s_{n-1}) < v(s_{n+1} - s_n)$, for every $n \in \mathbb{N}$.
The *breadth ideal* of E is:

$$\mathrm{Br}(E) = \{b \in K \mid v(b) > v(s_{n+1} - s_n), \forall n \in \mathbb{N}\}$$

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Note that $\mathrm{Br}(E) = \{0\}$ if and only if E is a Cauchy sequence (therefore E converges to its unique limit in the completion $\widehat{K}$). In general we have:

Lemma (Kaplansky)

Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$ be a pseudo-convergent sequence and $\alpha \in K$ a pseudo-limit. Then the set of pseudo-limits of E in K is $\{\alpha\} + \mathrm{Br}(E)$.

Precompactness is not necessary (after Loper and Werner)

Definition (Kaplansky)

Let $E = \{s_n\}_{n\in\mathbb{N}}$ be a pseudo-convergent sequence. We say that E is of *transcendental type* if $v(f(s_n))$ is definitively constant for all $f \in K[X]$. Otherwise we say that E is of *algebraic type*.

Remark: E is of algebraic type if some $\alpha \in \overline{K}$ is a pseudo-limit of E (with respect to some extension of v to $\overline{K}$).

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Theorem (Loper and Werner, 2016)

Let V be a valuation domain of rank one and let $E = \{s_n\}_{n \in \mathbb{N}} \subset V$ be a pseudo-convergent sequence. Then $\text{Int}(E, V)$ is Prüfer if and only if E is either of transcendental type or the breadth ideal $\mathrm{Br}(E) = (0)$.

Example

If $E = \{s_n\}_{n \in \mathbb{N}} \subset V$ is a pseudo-convergent sequence of transcendental type and non-zero breadth ideal, then $\text{Int}(E, V)$ is Prüfer and E is not precompact.

Pseudo-monotone sequences (after Chabert)

Let $V \subset K$ be a rank one valuation domain.

Definition

Let $E = \{s_n\}_{n\in\mathbb{N}} \subset K$. We say that E is a *pseudo-monotone sequence* if $\{v(s_{n+1} - s_n)\}_{n\in\mathbb{N}}$ is monotone, that is, one of the following conditions holds:

i) $v(s_{n+1} - s_n) < v(s_{n+2} - s_{n+1})$, $\forall n \in \mathbb{N}$ (*pseudo-convergent*).

ii) $v(s_n - s_m) = \gamma \in \Gamma_v$, for all $n \neq m \in \mathbb{N}$ (*pseudo-stationary*).

iii) $v(s_{n+1} - s_n) > v(s_{n+2} - s_{n+1})$, $\forall n \in \mathbb{N}$ (*pseudo-divergent*).

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Let $\alpha \in K$. We say that α is a *pseudo-limit* of E if:

i) $v(\alpha - s_n)$ is strictly increasing ($\Leftrightarrow v(\alpha - s_n) = v(s_{n+1} - s_n)$, $\forall n \in \mathbb{N}$).

ii) $v(\alpha - s_n)$ is stationary, equal to $\gamma \in \Gamma_v$, for all $n \in \mathbb{N}$.

iii) $v(\alpha - s_n)$ is strictly decreasing ($\Leftrightarrow v(\alpha - s_{n+1}) = v(s_{n+1} - s_n)$, $\forall n \in \mathbb{N}$).

Polynomial closure

Definition

Let $S \subseteq K$. The *polynomial closure* of S is:

$$\overline{S} = \{x \in K \mid \forall f \in \mathrm{Int}(S, V), f(x) \in V\}$$

In particular, $\mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V)$.

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Note that

$$\overline{S} = \{x \in K \mid \text{Int}(S, V) \subset W_x\}$$

where $W_x = \{\varphi \in K(X) \mid \varphi(x) \in V\}$.

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Theorem (Chabert, 2010)

Let V be a rank one valuation domain. Then, the family $\{\overline{S} \mid S \subseteq K\}$ forms a topology of closed sets in K.

We call this topology the *polynomial topology* on K.

Polynomial topology is weaker than v-adic topology

For $S \subseteq K$, let $\widetilde{S}$ be the v-adic closure. Since every polynomial function is continuous for the v-adic topology, we have:

$$S \subseteq \widetilde{S} \subseteq \overline{S}$$

In particular, the polynomial topology is weaker than the v-adic topology.

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In particular, the polynomial topology is weaker than the v-adic topology.

Examples

- If V is DVR with finite residue field, then $\widetilde{S} = \overline{S}$: polynomial topology coincides with v-adic topology (Gilmer-McQuillan).
- If V is non-discrete, then we may have $S \subseteq \widetilde{S} \subsetneq \overline{S}$!! For example, for $\alpha \in K$ and $\gamma \in \Gamma_v$,

  $$\mathring{B}(\alpha, \gamma) = \{x \in K \mid v(x - \alpha) > \gamma\}$$

  is v-adically closed but its polynomial closure is

  $$B(\alpha, \gamma) = \{x \in K \mid v(x - \alpha) \geq \gamma\}$$

A criterion for Prüfer domains

Theorem

Let D be an integrally closed domain and P a prime ideal of D. Then D_P is a valuation domain if and only if there is no valuation overring V of D centered in P such that the residue field of V is transcendental over the quotient field of D/P.

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Application: $\mathrm{Int}(S,V)$ is not Prüfer if and only if $\mathrm{Int}(S,V)\subset W\subset K(X)$ for some valuation domain W which lies over V and such that $V/M\subset W/M_W$ is a transcendental extension.

Residually transcendental extensions

Definition

Let $W \subset K(X)$ be a valuation domain which lies over $V \subset K$. We say that W is a **residually transcendental extension** of V if the residue field extension $V/M_V \subset W/M_W$ is transcendental.

Definition

Let $\alpha \in K$ and $\delta \in \mathbb{R}$. Then for all $f \in K[X]$, $f(X) = \sum_i a_i(X-\alpha)^i$ we consider the *monomial valuation*:

$$v_{\alpha,\delta}(f) = \inf_i \{v(a_i) + i\delta\}$$

We let $V_{\alpha,\delta} \subset K(X)$ be the associated valuation domain. Clearly, $V_{\alpha,\delta} \cap K = V$.

Lemma

$V_{\alpha,\delta}$ is residually transcendental over V if and only if $\delta \in \Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$.

Classification of residually transcendental extensions

Let $\overline{K}$ be a fixed algebraic closure and $\Gamma_{\overline{v}} = \Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem (Alexandru & Popescu, 1988)

Let $\mathcal{W} \subset K(X)$ be a residually transcendental extension of V. Then there exist $\alpha \in \overline{K}$, $\delta \in \Gamma_{\overline{v}}$ and and extension $\overline{W}$ of V to $\overline{K}$ such that $\mathcal{W} = \overline{W}_{\alpha,\delta} \cap K(X)$ ($\overline{W}_{\alpha,\delta} \subset \overline{K}(X)$ is the monomial valuation associated to $\overline{W}$).

We denote $V_{\alpha,\delta} = \overline{W}_{\alpha,\delta} \cap K(X)$. Note that $V_{\alpha,\delta}$ depends on the chosen extension $\overline{W}$ of V to $\overline{K}$!

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Theorem

Let $R \subseteq K[X]$ be an integrally closed domain with quotient field $K(X)$ and such that $D = R \cap K$ is a Prüfer domain with quotient field K. Then R is Prüfer if and only if there is no valuation overring V of D and $(\alpha, \delta) \in \overline{K} \times \Gamma_{\overline{v}}$ such that $R \subset V_{\alpha,\delta}$.

A sufficient condition

Recall that for $(\alpha,\gamma)\in K\times\mathbb{R}$, $V_{\alpha,\gamma}$ is the valuation domain of the valuation $v_{\alpha,\gamma}(f)=\inf_k\{v(a_k)+k\gamma\}$, if $f(X)=\sum_k a_k(X-\alpha)^k$.

Lemma (Chabert)

Let V be either non-discrete or with infinite residue field. Let $S\subseteq V$, $\alpha\in K$ and $\gamma\in\mathbb{R}$. Then $Int(S,V)\subset V_{\alpha,\gamma}$ if and only if $B(\alpha,\gamma)=\{x\in K\mid v(x-\alpha)\geq\gamma\}\subseteq\overline{S}$, the polynomial closure of S.

Remark: this result is false if V is a DVR with finite residue field, since $\mathrm{Int}(V)$ is Prüfer in that case.

Proposition (Chabert)

Let $S\subseteq V$. Let $E=\{s_n\}_{n\in\mathbb{N}}\subseteq S$ be a pseudo-monotone sequence with breadth $\gamma\in\mathbb{R}$ and a pseudo-limit $\alpha\in V$. Then $\mathrm{Int}(S,V)\subset V_{\alpha,\gamma}$.

A first necessary condition

For $(\alpha, \gamma), (\alpha, \gamma') \in K \times \mathbb{R}$, $V_{\alpha,\gamma}$, $V_{\alpha,\gamma'}$ are incomparable, but $V_{\alpha,\gamma} \cap K[X] \subseteq V_{\alpha,\gamma'} \cap K[X]$ if and only if $\gamma \leq \gamma'$.

Proposition

Let $S \subseteq V$ be a subset such that $\text{Int}(S, V) \subset V_{\alpha,\gamma'}$ for some $(\alpha, \gamma') \in V \times \Gamma_v$. Let $\gamma = \inf\{\gamma' \in \Gamma_v \mid \text{Int}(S, V) \subset V_{\alpha,\gamma'}\} \in \mathbb{R}$. Then $\text{Int}(S, V) \subset V_{\alpha,\gamma}$ and there exists a pseudo-monotone sequence $E = \{s_n\}_{n\in\mathbb{N}} \subseteq S$ with breadth γ and pseudo-limit which is equal either to α or to $\alpha + t$, with $v(t) = \gamma$.

Note that $V_{\alpha,\gamma} = V_{\alpha+t,\gamma}$, if $v(t) = \gamma$.

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Note that $V_{\alpha,\gamma}=V_{\alpha+t,\gamma}$, if $v(t)=\gamma$.

But this is not enough! Recall that $\mathrm{Int}(S,V)$ is not Prüfer if and only if there exists $(\alpha,\gamma)\in\overline{K}\times\Gamma_{\overline{v}}$ such that $\mathrm{Int}(S,V)\subset V_{\alpha,\gamma}$.
Thus, we have also to consider the case of $\alpha\in\overline{K}\setminus K$.

Reduction to the base field K

Proposition

Let S be a subset of an integrally closed domain D with quotient field K. Let F be an algebraic extension of K and D_F the integral closure of D in F. Then the integral closure of $\text{Int}(S, D)$ *in* $F(X)$ *is the ring* $\text{Int}(S, D_F)$.

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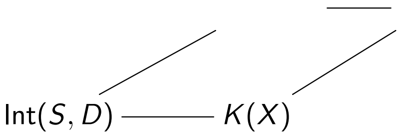

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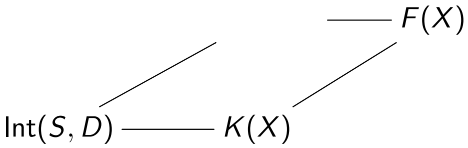

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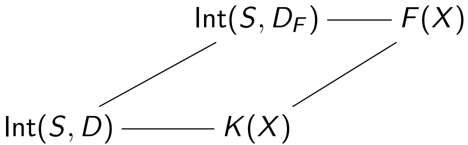

Reduction to the base field K

Note that $(\alpha,\gamma)\in\overline{K}\times\Gamma_{\overline{v}}\Rightarrow(\alpha,\gamma)\in F\times\Gamma_w$, for some finite extension F of K and a valuation w of F extending v.

Corollary

Let $(\alpha,\gamma)\in F\times\Gamma_w$, where F is a finite field extension of K and W is a valuation domain of F lying over V. If S is a subset of V such that $\mathrm{Int}(S,V)\subset V_{\alpha,\gamma}=W_{\alpha,\gamma}\cap K(X)$, then $\mathrm{Int}(S,W)\subset W_{\alpha,\gamma}$.

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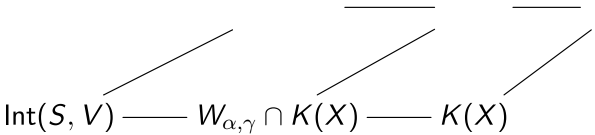

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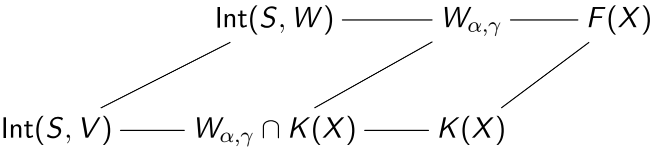

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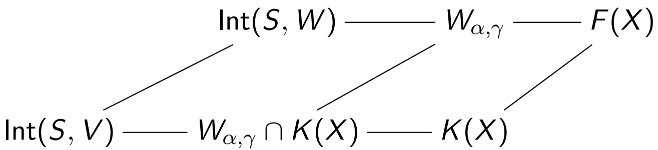

Note that the $\mathrm{Int}(S,V_F) = \bigcap_{i=1}^n \mathrm{Int}(S,W_i)$, where $V_F = \bigcap_{i=1}^n W_i$.

Final result

Theorem

Let $S \subseteq V$, V a rank one valuation domain. Then $\text{Int}(S, V)$ is Prüfer if and only if S does not contain a pseudo-monotone sequence $E = \{s_n\}_{n \in \mathbb{N}}$ which has a pseudo-limit $\alpha \in \overline{K}$ (with respect to some extension of v).

Remark: here, pseudo-limit is in the strict sense (i.e., a classical limit of a Cauchy sequence is not intended as a pseudo-limit).
Clearly, this result generalizes Loper and Werner's Theorem .

Discrete case

Corollary

Let V be a DVR and $S \subseteq V$. Then the following conditions are equivalent:

i) $\mathrm{Int}(S, V)$ *is Prüfer.*

ii) *there is no pseudo-stationary sequence contained in S.*

iii) S *is precompact.*

iv) *there is no* $(\alpha, \gamma) \in V \times \Gamma_v$ *such that* $\mathrm{Int}(S, V) \subset V_{\alpha,\gamma}$.

Thank you!