Perfectoid fields, deeply ramified fields and their relatives

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(joint work with Anna Blaszczok)

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In the following, p will always be the characteristic of the residue field.

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• Perfectoid fields caught our interest in connection with our work on the defect of valued field extensions.

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In the positive characteristic case, a perfect valued field (such as $\mathbb{F}_p((t))^{1/p^\infty}$) has no dependent defect extensions. What about perfectoid fields in mixed characteristic?

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where $\mathcal{O}_K$ is the valuation ring of K, $\mathcal{O}_{K^{\mathrm{sep}}}$ is the valuation ring of the separable-algebraic closure of K, and $\Omega_{B|A}$ denotes the module of relative differentials when A is a ring and B is an A-algebra.

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is surjective ("the Frobenius on $\mathcal{O}_K$ is surjective modulo p").

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Gdr fields and defect

We denote by $(vK)_{vp}$ the smallest convex subgroup of vK that contains vp if $\mathrm{char}\, K = 0$,

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Theorem

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Theorem

Take a valued field (K,v) with $\mathrm{char}\, Kv = p > 0$. Then (K,v) is a gdr field if and only if $(vK)_{vp}$ is p-divisible, Kv is perfect, and every separable defect extension of degree p is independent.

Tame extensions

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(T3) $[L : K] = (vL : vK)[Lv : Kv]$.

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Properties of tame fields

Tame fields have many good properties. For example:

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Valued function fields over separably tame fields have a relatively good structure theory. This is used to prove the above theorem, and it also has been applied to the problem of local uniformization.

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The important question arises whether results on tame fields can be generalized to deeply ramified or even gdr fields.

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A hierarchy of fields

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Questions of this type are very hard to study when the valued field under consideration does not have a p-divisible value group or does not have a perfect residue field.

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Preprints and further information

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html