Topological properties of subsets of the Zariski space

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The Zariski space and the Zariski topology

- We shall always consider a domain D and a field K containing D.
- $\mathrm{Zar}(K|D)$ is the set of valuation domains of K containing D.
  - If K is the quotient field of D, we set $\mathrm{Zar}(K|D) = \mathrm{Zar}(D)$, and its elements are the *valuation overrings* of D.
- The Zariski topology on $\mathrm{Zar}(K|D)$ is generated by the sets

$$\mathcal{B}(x_1, \ldots, x_n) := \{V \in \mathrm{Zar}(K|D) \mid x_1, \ldots, x_n \in V\}.$$

- Except trivial cases, the Zariski topology is not T_1 (i.e., not all points are closed); in particular, it is not Hausdorff.
  - The only closed points are the minimal elements.

Spectral spaces

- A topological space is spectral if it is homeomorphic to Spec(R) for some ring R.
  - Spectral spaces can be characterized topologically [Hochster, 1969].
- $\mathrm{Zar}(K|D)$ is a spectral space: more precisely, we can find (explicitly) an overring $\mathrm{Kr}(K|D)$ of $K[X]$ such that $\mathrm{Zar}(K|D) \simeq \mathrm{Spec}(\mathrm{Kr}(K|D))$.
- The constructible topology on a spectral space X is the coarsest topology where the open and compact subsets of X are both open and closed.
  - X^{cons} is a spectral space that is also Hausdorff.
  - The closed set of X^{cons} are spectral (in the starting topology).

Why the Zariski topology?

- Study of resolution of singularities (Zariski).
  - You need the compactness of $\mathrm{Zar}(K|D)$.
- Study of intersection of valuation rings.
  - If $X \subseteq \mathrm{Zar}(D)$ is a compact subset, each $V \in X$ is one-dimensional, and $\bigcap_{V\in X} \mathfrak{m}_V \neq (0)$, then $\bigcap_{V\in X} V$ is a one-dimensional Bézout domain (i.e., all finitely generated ideals are principal) [Olberding, 2017].
  - Study of holomorphy and real holomorphy rings.
- If D is a Prüfer domain, then $\mathrm{Zar}(D) \simeq \mathrm{Spec}(D)$.
  - If D is any domain, the map $\mathrm{Zar}(D) \longrightarrow \mathrm{Spec}(D)$, $V \mapsto \mathfrak{m}_V \cap D$ is a closed continuous surjection.
  - If D is any domain, the map $P \mapsto D_P$ embeds $\mathrm{Spec}(D)$ in the set of *overrings* of D, endowed with the Zariski topology.

Non-compact subspaces of valuation rings

- While $\mathrm{Zar}(D)$ is always compact, the same does not happen for subsets.
- Let V be a minimal element of $\mathrm{Zar}(D)$. If $\mathrm{Zar}(D) \setminus \{V\}$ is compact, then V is the integral closure of $D[x_1, \ldots, x_n]_M$ for some $x_1, \ldots, x_n \in K$ and $M \in \mathrm{Max}(D[x_1, \ldots, x_n])$.
    - The proof uses the integral closure of ideals and a criterion based on semistar operations.
- This cannot happen in the following cases:
    - D is Noetherian and $\dim(V) \geq 2$;
    - $\dim(V) > 2\dim(D)$;
    - D is local and $\bigcap\{P \mid P \in X\} = (0)$ for some family X of nonzero incomparable prime ideals.

When $\mathrm{Zar}(K|D)$ is Noetherian

- A topological space is *Noetherian* if all its subsets are compact.
  - If R is a Noetherian ring, $\mathrm{Spec}(R)$ is a Noetherian space.
- If $D = F$ is a field, then $\mathrm{Zar}(K|F)$ is a Noetherian space if and only if $\mathrm{trdeg}_F K \leq 1$ and, if $X \in K$ is transcendental over F, then every valuation on $F[X]$ extends to finitely many valuations of K.
- If D is local, then $\mathrm{Zar}(D)$ can be Noetherian only if D is a *pseudo-valuation domain* (PVD).
  - D is a PVD if its maximal ideal M is the maximal ideal of a valuation overring.
  - If $\mathrm{Zar}(D)$ is Noetherian, then $\mathrm{Zar}(D) \setminus \mathrm{Zar}_{\min}(D)$ is linearly ordered.
- $\mathrm{Zar}(D)$ is Noetherian if and only if $\mathrm{Spec}(D)$ and $\mathrm{Zar}(D_M)$ are Noetherian for every $M \in \mathrm{Max}(D)$.

Overrings of Noetherian domains

Let D be a Noetherian ring with $\dim(D) \geq 2$, and quotient field K.

- With the same methods as above, the space Δ of Noetherian valuation overrings of D is not compact.
- Consider the space $\mathrm{Over}(D)$ of *overrings* of D (i.e., rings between D and K) with the Zariski topology.
- The set of overrings of D that are Noetherian is compact (it has a minimum) but it is not a spectral space.
- None of the following is compact:
  - $\{T \in \mathrm{Over}(D) \mid T \text{ is a principal ideal domain}\}$;
  - $\{T \in \mathrm{Over}(D) \mid T \text{ is a Dedekind domain}\}$;
  - $\{T \in \mathrm{Over}(D) \mid T \text{ is Noetherian with } \dim(T) \leq 1\}$.

Pseudo-convergent sequences

From now on, let V be a one-dimensional valuation ring with nondiscrete valuation v, value group $\Gamma_v \subseteq \mathbb{R}$ and quotient field K.

We want to study subsets of $\mathrm{Zar}(K(X)|V)$.

- $E := \{s_n\}_{n\in\mathbb{N}} \subset K$ is a *pseudo-convergent sequence* if

$$v(s_n - s_{n-1}) < v(s_{n+1} - s_n)$$

  for all $n \in \mathbb{N}$, $n \geq 1$.
- We set $\delta_n := v(s_{n+1} - s_n)$: they form a strictly increasing sequence with limit $\delta \in \mathbb{R} \cup \{\infty\}$ (called the *breadth* of E).
- If $\delta = \infty$, then E is a Cauchy sequence.

Pseudo-limits

- $\alpha \in K$ is a *pseudo-limit* of E if $v(\alpha - s_n) < v(\alpha - s_{n+1})$ for all $n \in \mathbb{N}$.
- You can also consider pseudo-limits in $\overline{K}$ with respect to an extension u of v.
- Fix an extension u. The set of pseudo-limits of E in $\overline{K}$, if nonempty, is a ball of radius $e^{-\delta}$.
- We distinguish two types of pseudo-convergent sequences:
  - E is of algebraic if $v(f(s_n))$ is definitively increasing for some polynomial $f \in K[X]$;
  - E is of transcendental if $v(f(s_n))$ is definitively constant for all $f \in K[X]$.
- Equivalently, it is algebraic if E has pseudo-limits in $\overline{K}$, transcendental if it doesn't.
- If E is a Cauchy sequence, there is only one pseudo-limit in $\widehat{K}$, which is algebraic over K if E is algebraic and transcendental if E is transcendental.

Valuation domains associated to E (1)

- We want to associate to a pseudo-convergent sequence an extension of V to $K(X)$.
- Let $E := \{s_n\}_{n\in\mathbb{N}}$. For every $\phi \in K(X)$, let

$$w_E(\phi) := \lim_{n\to\infty} v(\phi(s_n))$$

  - If E is trascendent, w_E is a valuation.
  - If E is algebraic and $\delta_E < \infty$, w_E is a valuation.
  - If E is algebraic and Cauchy, w_E is a pseudo-valuation on $K[X]$.
- If w_E is a valuation, we denote by W_E its valuation ring.
- W_E is always one-dimensional and an extension of V.
- If K is algebraically closed, every rank-one extension of V to $K(X)$ is in the form W_E [Ostrowski, 1935].

Valuation domains associated to E (2)

- If $E = \{s_n\}_{n\in\mathbb{N}}$, we define

$$V_E := \{\phi \in K(X) \mid \phi(s_n) \in V \text{ for all large } n\}.$$

  - V_E was defined in [Loper and Werner, 2016] for E transcendental or when $\delta_E = \infty$.
- V_E is always an extension of V to $K(X)$.
- If E is of transcendental type, $V_E = W_E$ has rank 1.
- If E is of algebraic type, then:
  - if δ is torsion over Γ_v, then V_E has rank 2 and W_E has rank 1;
  - if δ is not torsion over Γ_v, then $V_E = W_E$ has rank 1;
  - if $\delta = \infty$, then V_E has rank 2, and its one-dimensional overring is $K[X]_{(q)}$, where q is the minimal polynomial of the limit of E.
- We can describe explicitly the valuation v_E.

Equivalence

- We say that E and F are *equivalent* if $\delta_E = \delta_F$ and, for every $k \in \mathbb{N}$, there are $i_0, j_0 \in \mathbb{N}$ such that, whenever $i \geq i_0$, $j \geq j_0$ then

$$v(s_i - t_j) > v(t_{k+1} - t_k).$$

  - This is a generalization of the concept of equivalence between Cauchy sequences.
- The following are equivalent:
  - E and F are equivalent;
  - $V_E = V_F$;
  - $W_E = W_F$ (when they are defined).
- If E and F are equivalent, then they are either both algebraic or both transcendental.
- If E and F are algebraic, then they are equivalent if and only if they have the same pseudo-limits in $\overline{K}$.

The space $\mathcal{W}$

Let

$$\mathcal{W} := \{W_E \mid E \text{ is a not algebraic and Cauchy}\} \subseteq \mathrm{Zar}(K(X)|V).$$

- The Zariski and the constructible topology agree on $\mathcal{W}$.
- $\mathcal{W}$ is regular (i.e., a point and a closed set can be separated by open sets).
- $\mathcal{W}$ is zero-dimensional (i.e., a point and a closed set can be separated by clopen sets).
- $\mathcal{W}$ is not compact.

The space $\mathcal{V}$

Let

$$\mathcal{V} := \{V_E \mid E \text{ is a pseudo-convergent sequence}\} \subseteq \mathrm{Zar}(K(X)|V).$$

- Each point of $\mathcal{V}$ is closed.
- $\mathcal{V}$ is regular in the Zariski topology.
- The Zariski and the constructible topology agree on $\mathcal{V}$ if and only if the residue field of V is finite.
- Endow $\mathcal{V}$ with the Zariski topology. The map

$$\begin{aligned} \mathcal{W} &\longrightarrow \mathcal{V} \\ W_E &\longmapsto V_E \end{aligned}$$

is continuous and injective, but *not* a topological embedding.

Fixed breadth (1)

Let

$$\mathcal{V}(\bullet,\delta) := \{V_E \in \mathcal{V} \mid \delta_E = \delta\}.$$

- If $E := \{s_n\}_{n\in\mathbb{N}}$, $F := \{t_n\}_{n\in\mathbb{N}}$, set

$$d_\delta(V_E, V_F) := \lim_{n\to\infty} \max\{d(s_n, t_n) - e^{-\delta}, 0\}.$$

- d_δ is an ultrametric distance of $\mathcal{V}(\bullet,\delta)$.
- If $E, F \in \mathcal{V}(\bullet,\delta)$ are algebraic, $V_E \neq V_F$, and α_E, α_F are pseudo-limits, then

$$d_\delta(V_E, V_F) = d(\alpha_E, \alpha_F) - e^{-\delta} = e^{-v(\alpha_E-\alpha_F)} - e^{-\delta}$$

- $\mathcal{V}(\bullet,\infty)$ is essentially $\widehat{K}$, and d_∞ reduces to the distance induced by $\widehat{v}$.

Fixed breadth (2)

- $\mathcal{V}(\bullet, \delta)$ is complete under d_δ.
- An important *dense* subset is

$$\mathcal{V}_K(\bullet, \delta) := \{V_E \in \mathcal{V}(\bullet, \delta) \mid E \text{ has a pseudo-limit in } K\}.$$

  - $\mathcal{V}_K(\bullet, \infty)$ correspond to K.
- The Zariski topology, the constructible topology and the topology induced by d_δ coincide on $\mathcal{V}(\bullet, \delta)$.
  - In particular, the two topologies agree on each slice, but do not agree (in general) on the whole $\mathcal{V}$.
- The various d_δ *cannot* be unified to a metric on the whole $\mathcal{V}$.
  - $\mathcal{V}(\bullet, \delta)$ is not closed!

Fixed center

Take $\beta \in \overline{K}$, and a fixed extension u of v to $\overline{K}$. Let

$$\mathcal{V}^u(\beta, \bullet) := \{V_E \in \mathcal{V} \mid \beta \text{ is a pseudo-limit of } E \text{ wrt } u\}.$$

- $\mathcal{V}^u(\beta, \bullet)$ is closed in $\mathcal{V}$.
- The Zariski and the constructible topology agree on $\mathcal{V}^u(\beta, \bullet)$.
- Two valuation domains in $\mathcal{V}^u(\beta, \bullet)$ are distinguished by the breadth; hence, we want to consider the map

$$\begin{aligned} \mathcal{V}^u(\beta, \bullet) &\longrightarrow (-\infty, +\infty] \\ V_E &\longmapsto \delta_E \end{aligned}$$

- Its range is $(-\infty, \delta(\beta, K)]$, where $\delta(\beta, K) := \sup\{u(\beta - x) \mid x \in K\}$ depends on the distance between β and K.
- What about the topological level?

Fixed center (2)

- On $(-\infty, \delta(\beta, K)]$, we put the topology generated by the sets $(a, b]$, with $b \in \mathbb{Q}\Gamma_v$.
  - This is a variant of the upper limit topology.
- Under this topology, the map $V_E \mapsto \delta_E$ becomes a homeomorphism.
- The following are equivalent:
  - $\mathcal{V}^u(\beta, \bullet)$ is metrizable;
  - there is an ultrametric distance on $\mathcal{V}^u(\beta, \bullet)$;
  - $\mathcal{V}^u(\beta, \bullet)$ is second-countable;
  - Γ_v is countable.
- If Γ_v is not countable, then $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ is not metrizable.

Open problems

- Is $\mathcal{V}$ zero-dimensional?
- If Γ_v is countable, are $\mathcal{V}$ and $\mathcal{W}$ metrizable?
- If Γ_v is countable, is $\mathrm{Zar}(K(X)|V)^{\mathrm{cons}}$ metrizable?
- More generally, when is $\mathrm{Zar}(K|D)^{\mathrm{cons}}$ metrizable?
- If any of them is metrizable, can we find an *ultrametric* distance?

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