Extensions of valuations to the Henselization and completion

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K a field with a valuation ν
Φ_ν value group
V_ν valuation ring with maximal ideal m_ν
(R, m_R) local domain with QF K

Semigroup: $S^R(\nu) = \{\nu(f) \mid f \in R \setminus \{0\}\}$

Associated graded ring of R along ν:

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma \in \Phi_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R) = \bigoplus_{\gamma \in S^R(\nu)} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R)$$

$$\mathcal{P}_\gamma = \{f \in R \mid \nu(f) \geq \gamma\}, \;\; \mathcal{P}^+_\gamma(R) = \{f \in R \mid \nu(f) > \gamma\}$$

Question 1

Suppose R is a Noetherian (excellent) local domain which is dominated by a valuation ν. Does there exist a regular local ring R' of the quotient field K of R such that ν dominates R' and R' dominates R, a prime ideal P of the m_R-adic completion $\widehat{R'}$ such that $P \cap R' = (0)$ and an extension $\hat{\nu}$ of ν to the QF of $\widehat{R'}/P$ which dominates $\widehat{R'}/P$ such that

$$\mathrm{gr}_{\nu}(R') \cong \mathrm{gr}_{\hat{\nu}}(\widehat{R'}/P)?$$

$$
\begin{array}{ccc}
V_\nu & \rightarrow & V_{\hat\nu}\\
\uparrow & & \uparrow\\
R' & \rightarrow & \widehat{R'}/P\\
\uparrow & & \\
R & &
\end{array}
$$

If ν has rank 1, then setting

$$P(\hat{R})_\infty = \{f \in \hat{R} \mid \nu(f) = \infty\}$$

we have

$$\mathrm{gr}_\nu(R) \cong \mathrm{gr}_{\hat{\nu}}(\hat{R}/P(\hat{R})_\infty).$$

so if ν has rank 1, then Question 1 has a positive answer for local domains R and rank 1 valuations ν which admit local uniformization. If R is essentially of finite type over a field of char. 0, then we can even take P so that $\widehat{R'}/P$ is a regular local ring.

Question 2

Suppose R is a Noetherian (excellent) local domain which is dominated by a valuation ν. Does there exist a regular local ring R' of the quotient field K of R such that ν dominates R' and R' dominates R, and an extension ν^h of ν to the QF of the Henselization $(R')^h$ of R' which dominates $(R')^h$ such that

$$\mathrm{gr}_{\nu}(R') \cong \mathrm{gr}_{\hat{\nu}}((R')^h)?$$

$$\begin{array}{ccc} V_\nu & \rightarrow & V_{\nu^h} \\ \uparrow & & \uparrow \\ R' & \rightarrow & (R')^h \\ \uparrow & & \\ R & & \end{array}$$

If Question 1 is true then so is Question 2. A start on answering Question 2 is the following proposition.

Proposition 3 [C] Suppose R and S are normal local rings such that R is excellent, S lies over R and S is unramified over R, $\tilde{\nu}$ is a valuation of the QF L of S which dominates S and ν is the restriction of $\tilde{\nu}$ to the QF K of R. Suppose L is finite over K. Then there exists a normal local ring R' of K which is dominated by ν and dominates R' such that if R'' is a normal local ring of K which is dominated by ν and dominates R', S'' is the normal local ring of L which is dominated by $\tilde{\nu}$ and lies over R'', then $R'' \rightarrow S''$ is unramified and

$$\mathrm{gr}_{\hat{\nu}}(S'') \cong \mathrm{gr}_{\nu}(R'') \otimes_{R''/m_{R''}} S''/m_{S''}.$$

$$\begin{array}{ccc}
V_\nu & \rightarrow & V_{\tilde\nu}\\
\uparrow & & \uparrow\\
R'' & \rightarrow & S''\\
\uparrow & & \\
R' & & \\
\uparrow & & \\
R & \rightarrow & S
\end{array}$$

Questions 1 and 2 have a negative answer in general (even in equicharacteristic 0).

Theorem 4

Suppose k is an algebraically closed field. Then there exists a 3 dimensional regular local ring T_0, which is a localization of a finite type k-algebra, with residue field k, and a valuation φ of the quotient field K of T_0 which dominates T_0 and whose residue field is k, such that if T is a regular local ring of K which is dominated by φ and dominates T_0, T^h is the Henselization of T and φ^h is an extension of φ to the quotient field of T^h which dominates T^h, then

$$S^{T^h}(\varphi^h) \neq S^T(\varphi)$$

under the natural inclusion $S^T(\varphi) \subset S^{T^h}(\varphi^h)$.

Theorem 4 gives a counterexample to Questions 1 and 2.

$T \subset T^h \subset \hat{T}/P$ if $P \cap T = (0)$

Outline of proof of Theorem 4

$$R_0 = k[x, y, t]_{(x,y)} \cong k(t)[x, y]_{(x,y)}$$

Define a valuation ν dominating R_0 by constructing a generating sequence

$$P_0 = x, P_1 = y, P_2, \ldots$$

Let $\overline{p}_1, \overline{p}_2, \ldots$ be the sequence of prime numbers, excluding the characteristic of k. Define

$$a_1 = \overline{p}_1 + 1$$

and inductively define a_i by

$$a_{i+1} = \overline{p}_i \overline{p}_{i+1} a_i + 1.$$

Define

$$P_{i+1} = P_i^{\overline{p}_i^2} - (1+t)x^{\overline{p}_i a_i}$$

for $i \geq 1$. Set $\nu(x) = 1$, $\nu(P_i) = \frac{a_i}{\overline{p}_i}$ for $i \geq 1$.

$$\Phi_\nu = \cup_{i\geq 1} \frac{1}{\overline{p}_1 \overline{p}_2 \cdots \overline{p}_i}\mathbb{Z}$$

Let $\overline{k}$ be an algebraic closure of $k(t)$, and $\alpha_i \in \overline{k}$ be a root of

$$f_i(u) = u^{\overline{p}^i} - (1+t) \in k[u]$$

for $i \geq 1$. $f_i(u)$ is the minimal polynomial of α_i over $k(\alpha_1, \ldots, \alpha_{i-1})$.

$$V_\nu/m_\nu = k(\{\alpha_i \mid i \geq 1\}) = k[\{(1+t)^{\frac{1}{\overline{p}_i}} \mid i \geq 1\}]$$

$$\alpha_i = \left[\frac{P_i^{\overline{p}_i}}{x^{a_i}}\right]$$

Suppose A is a regular local ring of the QF K of R_0 which is dominated by ν and dominates R_0. Then there exists a generating sequence of ν in A

$$Q_0 = u, Q_1 = w, Q_2 = w^{\overline{p}c} - (1+t)\tau^{\overline{p}} z^{\overline{p}e}, \ldots$$

where $\overline{p} = \overline{p}_{1+l}$ for some l and τ is a unit in A.

Let λ be a $\overline{p}$-th root of $1+t$ in an algebraic closure of K, $L = K(\lambda)$ and $\overline{\nu}$ be an extension of ν to L. Let $\varepsilon \in k$ be a primitive $\overline{p}$-th root of unity.

Let $B = A[\lambda]$, $C = B_{m_{\overline{\nu}} \cap B}$. $A \to C$ is unramifed, so C is a regular local ring with regular parameters z, w.

Proposition 5 $S^C(\overline{\nu}) \neq S^A(\nu)$.

proof: $S^A(\nu) = S(\{\nu(Q_i) \mid i \geq 0\})$.

$$Q_0 = z, Q_1 = w, Q_2 = w^{\overline{p}c} - (1+t)\tau^{\overline{p}} z^{\overline{p}e}, \ldots$$

$$\gamma_1 = \left[\frac{w^c}{z^e}\right] \in V_\nu/m_\nu \subset V_{\overline{\nu}}/m_{\overline{\nu}}$$

Let

$$0 \neq \beta = [\lambda\tau] \in V_{\overline{\nu}}/m_{\overline{\nu}}.,$$

$$h_j = w^c - \varepsilon^j \lambda\tau z^e \in C.$$

If $\varepsilon^j\beta \neq \gamma$, then $\overline{\nu}(h_j) = e\nu(z)$,

$$\sum_{j=1}^{\overline{p}} \overline{\nu}(h_j) = \nu(Q_2) > \overline{p}e\nu(z)$$

implies there exists a unique value of j such that $\varepsilon^j\beta = \gamma_1$ and $\overline{\nu}(h_j) > e\nu(z)$. If $\overline{\nu}(h_j) \in S^A(\nu)$, then

$$\overline{\nu}(h_j) \in S(\nu(z), \nu(w))$$

since

$$\overline{\nu}(h_j) = \nu(Q_2) - (\overline{p} - 1)e\nu(z) < \nu(Q_2).$$

Thus $\nu(Q_2) \in G(\nu(z), \nu(w))$, a contradiction.

Let μ be a valuation of

$$V_\nu/m_\nu = k(t)[\{1+t)^{\frac{1}{p_i}} \mid i \ge 1\}]$$

which is an extension of the (t)-adic valuation on $k[t]_{(t)}$. The value group of μ is $\mathbb{Z}$. Let φ be the composite valuation of ν and μ on K, so that $V_\varphi = \pi^{-1}(V_\mu)$, where $\pi : V_\nu \to V_\nu/m_\nu$.

$$V_\varphi/m_\varphi = V_\mu/m_\mu = k.$$

Let $T_0 = k[t,x,y]_{(t,x,y)}$ which is dominated by φ. Proposition 5 implies

Proposition 6

Suppose T is a regular local ring of K which dominates T_0 and is dominated by φ. Then there exists a finite separable extension field L of K such that T is unramified in L. Further, if $\overline{\varphi}$ is an extension of φ to L and if U is the normal local ring of L which lies over T and is dominated by $\overline{\varphi}$, then

1) U is a regular local ring
2) $T \rightarrow U$ is unramified with no residue field extension
3) $S^U(\overline{\varphi}) \neq S^T(\varphi)$

Proof of Theorem 4:

Construction of T^h (after Nagata). Let N be a separable closure of K. N is an (infinite) Galois extension of K with Galois group $G(N/K)$. Let E be a local ring of the integral closure of T in N.

$$G^s(E/T) = \{\sigma \in G(N/K) \mid \sigma(E) = E\}.$$

$$T^h = E^{G^s(E/T)}$$

with QF $M = N^{G^s(E/T)}$.

Let $K \to L$ be the field extension of Proposition 6. Choose an embedding $K \to L \to N$. Let U be the local ring of the integral closure of T in L which is dominated by E. U is unramifed over T with no residue field extension, so $L \subset M$ and U is dominated by T^h. Let $\overline{\varphi} = \varphi^h | L$. Then $\overline{\varphi}$ dominates U and T^h dominates U, so $S^U(\overline{\varphi}) \subset S^{T^h}(\varphi^h)$. But $S^U(\overline{\varphi}) \neq S^T(\varphi)$ by Proposition 6, so $S^{T^h}(\varphi^h) \neq S^T(\varphi)$.