UNIVERSALITY OF THE RIEMANN ZETA-FUNCTION IN SHORT INTERVALS

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In 1975, S.M. Voronin discovered the universality of the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$, on the approximation of non-vanishing analytic functions defined in the strip $D=\left\{s\in\mathbb{C}:\frac{1}{2}<\sigma<1\right\}$ by shifts $\zeta(s+i\tau)$, $\tau\in\mathbb{R}$. Denote by $\mathcal{K}$ the class of compact subsets of D with connected complements, and by $H_0(K)$, $K\in\mathcal{K}$, the class of continuous non-vanishing functions on K that are analytic in the interior of K. Then a modern version of the Voronin theorem says that if $K\in\mathcal{K}$ and $f(s)\in H_0(K)$, then, for every $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

In the report, we will discuss versions of the above theorem in short intervals.

Theorem 1. *Suppose that $T^{1/3}(\log T)^{26/15}\leqslant H\leqslant T$. Let $K\in\mathcal{K}$ and $f(s)\in H_0(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{H}\operatorname{meas}\left\{\tau\in[T,T+H]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

Theorem 1 has the following modification.

Theorem 2. *Under hypotheses of Theorem 1, the limit*

$$\lim_{T\to\infty}\frac{1}{H}\operatorname{meas}\left\{\tau\in[T,T+H]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0$$

exists for all but at most countably many $\varepsilon>0$.

The motivation of the universality in short intervals is related to the effectivization problem of universality theorems: given a function f, set K and ε, to indicate an interval as short as possible containing $\tau>0$ such that

$$\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon.$$